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$N = 2$ Einstein-Yang-Mills's BPS solutions

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Abstract: We find the general form of all the supersymmetric configurations and solutions of $N = 2, d = 4$ Einstein-Yang-Mills theories. In the timelike case, which we study in great detail, giving many examples, the solutions to the full supergravity equations can be constructed from known flat spacetime solutions of the Bogomol'nyi equations. This allows the regular supersymmetric embedding in supergravity of regular monopole solutions ('t Hooft-Poyakov's, Weinberg's, Wilkinson and Bais's) but also embeddings of irregular solutions to the Bogomol'nyi equations which turn out to be regular black holes with different forms of non-Abelian hair once the non-triviality of the spacetime metric is taken into account. The attractor mechanism is realized in a gauge-covariant way.
In the null case we determine the general equations that supersymmetric configurations and solutions must satisfy but we do not find relevant new supersymmetric solutions.

Keywords: Black Holes in String Theory, Supergravity Models.

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Contents

1. Introduction

Supersymmetric solutions of supergravity theories are playing a crucial rôle in may of the developments that Superstring Theory has seen in the last few years. The knowledge of all the possible solutions can lead to new interesting models from which we can learn more about the possible vacua of the theory, their potential holographic relations with CFTs etc. Achieving a complete characterization and classification of all the supersymmetric solutions of supergravity theories is, thus, an important goal with may potential spin-offs.

Most of the work done so far in this subject has been focussed on higher-dimensional ungauged theories. The 4-dimensional theories are equally interesting, though, since they admit solutions such as the much-studied families of charged extreme black holes found in refs. [1] in ungauged $N = 4, d = 4$ supergravity and in refs. [2, 3] in $N = 2, d = 4$ ungauged supergravity coupled to vector multiplets.

The systematic study and classification of supersymmetric solutions of 4-dimensional supergravities was pioneered 25 years ago by Tod 25 in ref. [4], in which he completely solved the problem in pure, ungauged, $N = 2, d = 4$ supergravity. Apart from another work on $N = 4, d = 4$ supergravity [5], the subject was not reanimated until quite recently: the problem was solved for pure, gauged $N = 2, d = 4$ supergravity in refs. [6−8], for ungauged $N = 2, d = 4$ supergravity coupled to vector supermultiplets in ref. [9], and for the same theory with a $U(1)$ gauging in ref. [10]; ungauged $N = 2, d = 4$ supergravity coupled to vector supermultiplets and hypermultiplets was dealt with in ref. [11]. Finally, the problem was solved for pure, ungauged $N = 4, d = 4$ supergravity in [12] and for matter-coupled $N = 1, d = 4$ supergravity in ref. [13] for the ungauged case (without superpotential but with non-trivial kinetic matrix) and in ref. [14] for the gauged case with superpotential but without kinetic matrix.

The cases considered so far (see above) only include non-Abelian gauge groups in the $N = 1, d = 4$ case, which does not admit supersymmetric black-hole-type nor static monopole-like solutions: they can only exist in $N > 1, d = 4$ theories. We are, therefore, led to consider $N > 1, d = 4$ theories with non-Abelian gaugings. Some interesting non-Abelian monopole solutions are known in gauged $N = 4, d = 4$ supergravity (namely, the Chamseddine-Volkov monopole [15]) and similar solutions must exist in $N = 2, d = 4$ theories with non-Abelian gaugings, of which there is a much wider variety.

In this paper we are going to study the classification of the supersymmetric solutions of $N = 2, d = 4$ supergravity coupled to vector multiplets with non-Abelian gaugings of the special-Kähler manifold (that we will call $N = 2, d = 4$ super-Einstein-Yang-Mills (SEYM) theories for short) with the aim of finding non-Abelian generalizations of the known supersymmetric extreme black holes of the ungauged theories [3] and supersymmetric embeddings of YM monopoles in supergravity. We will not obtain a full classification in the strict sense but rather in the loose sense as used in the literature: we will resolve the problem into its simplest constituents, namely a finite number of functions and differential equations they need to satisfy; having solved these differential equations there is a systematic and straightforward way of reconstructing the supergravity solution.

As a by-product we will give a recipe (see section 4.5) which allows the embedding into $N = 2, d = 4$ supergravity of virtually any solution to the Bogomol'nyi equation [16]. This will allow us to construct several explicit examples of non-Abelian monopoles and non-Abelian black-holes.[1]

While the existence of the monopoles was expected due to the existence of the globally regular Chamseddine-Volkov [15] and Harvey-Liu [19] monopole solutions, the existence of regular extreme black-holes with non-trivial non-Abelian hair is a bit more surprising given the existence of a *non-Abelian baldness theorem* [20][2] that states that all the regular black-hole solutions of the $SU(2)$ Einstein-Yang-Mills theory with colour charges are actually embeddings of solutions with Abelian charges. The truly non-Abelian solutions of the EYM theory (the Bartnik-McKinnon particle [22] and its black hole generalizations [23]), which are only known numerically, do not have any asymptotic gauge charges. By contrast, some of our solutions, which are fully analytical, do have genuinely non-Abelian charges at infinity. Some of our solutions also have non-Abelian hair that does not result into any gauge charges at infinity. It is evident that the non-Abelian baldness theorems do not apply to $N = 2, d = 4$ SEYM theories, which have a different matter content, one in which the scalars play a prominent rôle.

One of the most interesting aspects of the supersymmetric black holes of ungauged $N = 2, d = 4$ supergravity is the existence of the *attractor mechanism* for the values of the scalars [2, 24]: independently of their asymptotic values, the values of the scalars on the event horizon are fully determined by the conserved charges. As a result, the Bekenstein-Hawking entropy depends only on conserved charges which is, by itself, a strong indication that it admits a microscopic interpretation. It is of the utmost interest, then, to study if and how the attractor mechanism works for the supersymmetric non-Abelian black holes in these theories. Our answer will be positive in a properly generalized way.

The plan of this article is as follows: in section 2 we will review gauged $N = 2$ $d = 4$ supergravity without hypermultiplets (to which we shall refer as $N = 2, d = 4$ SEYM), leaving information about isometries in Special Geometry and their implementation in supergravity for the appendix A. In section 3 we shall discuss the generic characteristics of the supersymmetric solutions, such as the Killing Spinor Equations and their implications for the equations of motion. In section 4, we shall characterize the solutions in the timelike case obtaining the minimal set of equations that need to be solved in order to have supersymmetric solutions to $N = 2$ $d = 4$ SEYM. Section 4.5 contains the step-by-step procedure to construct supersymmetric solutions of the theory starting with a solution of the Bogomol'nyi equations on $\mathbb{R}^3$ and which we will use in section 5 to construct and study different examples of solutions belonging to this class. These solutions split up into globally regular monopoles and black holes. Appendices B and C contain some complementary information needed for section 5. In section 6 we solve the null case. A discussion of our results and our conclusions are contained in section 7.

[1] A few examples have been published in refs. [17, 18].

[2] See the review paper [21] for further on this subject.

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2. Gauged $N = 2, d = 4$ supergravity coupled to vector supermultiplets

In this section we shall describe the action, equations of motion and supersymmetry transformation rules of gauged $N = 2, d = 4$ supergravity coupled to vector multiplets. In order to make this description brief, we only discuss the differences with the ungauged case, which is described in detail in ref. [11]. Some definitions and formulae related to the gauging of holomorphic isometries of special Kähler manifolds are contained in appendix A. We also refer the reader to ref. [25], the review ref. [26], and the original works refs. [27, 28] for more information.

The action restricted to the bosonic fields of these theories is

$$\begin{aligned} S = \int d^4x\sqrt{|g|}\,\big[R &+ 2\mathcal{G}_{ij^*}\mathfrak{D}_\mu Z^i \mathfrak{D}^\mu Z^{*\,j^*} + 2\Im\mathrm{m}\mathcal{N}_{\Lambda\Sigma}F^{\Lambda\,\mu\nu}F^{\Sigma}{}_{\mu\nu} \\ &-2\Re\mathrm{e}\mathcal{N}_{\Lambda\Sigma}F^{\Lambda\,\mu\nu\star}F^{\Sigma}{}_{\mu\nu} - V(Z,Z^*)\big]\ , \end{aligned} \tag{2.1}$$

where the potential $V(Z, Z^*)$ is given by

$$V(Z,Z^*) = 2\mathcal{G}_{ij^*}W^i W^{*j^*}\ , \tag{2.2}$$

with

$$W^i \equiv \frac{1}{2} g\mathcal{L}^{*\,\Lambda}k_\Lambda{}^i\,. \tag{2.3}$$

In these expressions g is the gauge coupling constant, the $k_\Lambda{}^i(Z)$ are holomorphic Killing vectors of $\mathcal{G}_{ij^*}$ and $\mathfrak{D}$ is the gauge covariant derivative (also Kähler-covariant when acting on fields of non-trivial Kähler weight) and is defined in appendix A.

This is not the most general gauged $N = 2, d = 4$ supergravity: if the $\mathfrak{sp}(2\bar{n})$ matrices $\mathcal{S}_\Lambda$ that provide a representation of the Lie algebra of the gauge group G_V, see eq. (A.26), are written in the form

$$\mathcal{S}_\Lambda = \begin{pmatrix} a_\Lambda{}^\Omega{}_\Sigma & b_\Lambda{}^{\Omega\Sigma} \\ \\ c_{\Lambda\Omega\Sigma} & d_{\Lambda\Omega}{}^\Sigma \end{pmatrix}\ , \tag{2.4}$$

then we are considering only those cases in which $b = 0$, so that only symmetries of the action are gauged, and $c = 0$. This last restriction is only made for the sake of simplicity as theories in which symmetries with $c \neq 0$ are gauged have complicated Chern-Simons terms.

Within this restricted class of theories, then, we can use eqs. (A.45) and (A.47) to rewrite the potential as

$$V(Z,Z^*) = \frac{1}{2}g^2 f^{*\Lambda\,i}f^{\Sigma}{}_i\mathcal{P}_\Lambda\mathcal{P}_\Sigma = -\frac{1}{4}g^2(\Im\mathrm{m}\mathcal{N})^{-1|\Lambda\Sigma}\mathcal{P}_\Lambda\mathcal{P}_\Sigma\,. \tag{2.5}$$

Then, since $\Im\mathrm{m}\mathcal{N}_{\Lambda\Sigma}$ is negative definite and the momentum map is real, the potential is positive semi-definite $V(Z, Z^*) \geq 0$. For constant values of the scalars $V(Z, Z^*)$ behaves as a non-negative cosmological constant $\Lambda = V(Z, Z^*)/2$ which leads to Minkowski ($\Lambda = 0$) or dS ($\Lambda > 0$) vacua. The latter cannot be maximally supersymmetric, however.

For convenience, we denote the bosonic equations of motion by

$$\mathcal{E}_a{}^\mu \equiv -\frac{1}{2\sqrt{|g|}}\frac{\delta S}{\delta e^a{}_\mu}\,, \qquad \mathcal{E}^i \equiv -\frac{\mathcal{G}^{ij^*}}{2\sqrt{|g|}}\frac{\delta S}{\delta Z^{*j^*}}\,, \qquad \mathcal{E}_\Lambda{}^\mu \equiv \frac{1}{8\sqrt{|g|}}\frac{\delta S}{\delta A^\Lambda{}_\mu}\,. \tag{2.6}$$

and the Bianchi identities for the vector field strengths by

$$\mathcal{B}^{\Lambda\,\mu} \equiv \mathfrak{D}_\nu \star F^{\Lambda\,\nu\mu}\,, \quad \star\mathcal{B}^\Lambda \equiv -\mathfrak{D}F^\Lambda\,. \tag{2.7}$$

Then, using the action eq. (2.1), we find

$$\begin{aligned}
\mathcal{E}_{\mu\nu} &= G_{\mu\nu} + 2\mathcal{G}_{ij^*}\left[\mathfrak{D}_\mu Z^i \mathfrak{D}_\nu Z^{*\,j^*} - \frac{1}{2}g_{\mu\nu}\mathfrak{D}_\rho Z^i \mathfrak{D}^\rho Z^{*\,j^*}\right] \\
&\quad + 8\Im\mathrm{m}\mathcal{N}_{\Lambda\Sigma}F^{\Lambda\,+}{}_\mu{}^\rho F^{\Sigma\,-}{}_{\nu\rho} + \frac{1}{2}g_{\mu\nu}V(Z,Z^*)\,, \qquad (2.8)\\
\mathcal{E}_\Lambda{}^\mu &= \mathfrak{D}_\nu \star F_\Lambda{}^{\nu\mu} + \frac{1}{2}g\Re\mathrm{e}(k_{\Lambda\,i^*}\mathfrak{D}^\mu Z^{*i^*})\,, \qquad (2.9)\\
\mathcal{E}^i &= \mathfrak{D}^2 Z^i + \partial^i \tilde{F}_\Lambda{}^{\mu\nu} \star F^\Lambda{}_{\mu\nu} + \frac{1}{2}\partial^i V(Z,Z^*)\,. \qquad (2.10)
\end{aligned}$$

In differential-form notation, the Maxwell equation takes the form

$$-\star\hat{\mathcal{E}}_\Lambda = \mathfrak{D}F_\Lambda - \frac{1}{2}g\star\Re\mathrm{e}\,(k^*_{\Lambda\,i}\mathfrak{D}Z^i)\,. \tag{2.11}$$

For vanishing fermions, the supersymmetry transformation rules of the fermions are

$$\begin{aligned}
\delta_\epsilon \psi_{I\,\mu} &= \mathfrak{D}_\mu \epsilon_I + \epsilon_{IJ}T^+{}_{\mu\nu}\gamma^\nu\epsilon^J\,, \qquad (2.12)\\
\delta_\epsilon \lambda^{Ii} &= i\,\not{\mathfrak{D}}Z^i\epsilon^I + \epsilon^{IJ}[\not{G}^{i\,+} + W^i]\epsilon_J\,, \qquad (2.13)
\end{aligned}$$

where $\mathfrak{D}_\mu\epsilon_I$ is given in eq. (A.39).

The supersymmetry transformations of the bosons are the same as in the ungauged case

$$\begin{aligned}
\delta_\epsilon e^a{}_\mu &= -\frac{i}{4}(\bar{\psi}_{I\,\mu}\gamma^a\epsilon^I + \bar{\psi}^I{}_\mu\gamma^a\epsilon_I)\,, \qquad (2.14)\\
\delta_\epsilon A^\Lambda{}_\mu &= \frac{1}{4}(\mathcal{L}^{\Lambda\,*}\epsilon^{IJ}\bar{\psi}_{I\,\mu}\epsilon_J + \mathcal{L}^\Lambda\epsilon_{IJ}\bar{\psi}^I{}_\mu\epsilon^J) \\
&\quad + \frac{i}{8}(f^\Lambda{}_i\epsilon_{IJ}\bar{\lambda}^{Ii}\gamma_\mu\epsilon^J + f^{\Lambda *}{}_{i^*}\epsilon^{IJ}\bar{\lambda}_I{}^{i^*}\gamma_\mu\epsilon_J)\,, \qquad (2.15)\\
\delta_\epsilon Z^i &= \frac{1}{4}\bar{\lambda}^{Ii}\epsilon_I\,. \qquad (2.16)
\end{aligned}$$

3. Supersymmetric configurations: general setup

Our first goal is to find all the bosonic field configurations $\{g_{\mu\nu}, F^\Lambda{}_{\mu\nu}, Z^i\}$ for which the Killing spinor equations (KSEs):

$$\begin{aligned}
\delta_\epsilon \psi_{I\,\mu} &= \mathfrak{D}_\mu \epsilon_I + \epsilon_{IJ}T^+{}_{\mu\nu}\gamma^\nu\epsilon^J = 0\,, \qquad (3.1)\\
\delta_\epsilon \lambda^{Ii} &= i\not{\mathfrak{D}}Z^i\epsilon^I + \epsilon^{IJ}[\not{G}^{i\,+} + W^i]\epsilon_J = 0\,, \qquad (3.2)
\end{aligned}$$

admit at least one solution.

Our second goal will be to identify among all the supersymmetric field configurations those that satisfy all the equations of motion (including the Bianchi identities).

Let us initiate the analysis of the KSEs by studying their integrability conditions.

3.1 Killing Spinor Identities (KSIs)

The off-shell equations of motion of the bosonic fields of bosonic supersymmetric configurations satisfy certain relations known as *Killing spinor identities* (KSIs) [29, 30]. If we assume that the Bianchi identities are always identically satisfied everywhere, the KSIs only depend on the supersymmetry transformation rules of the bosonic fields. These are identical for the gauged and ungauged theories, implying that their KSIs are also identical. If we do not assume that the Bianchi identities are identically satisfied everywhere, then they also occur in the KSIs, which now have to be found via the integrability conditions of the KSEs. In the ungauged case they occur in symplectic-invariant combinations, as one would expect, and take the form [9]

$$\mathcal{E}_a{}^{\mu}\gamma^a\epsilon^I - 4i\epsilon^{IJ}\langle\, \mathcal{E}^{\mu} \mid \mathcal{V}\,\rangle\epsilon_J = 0\,, \tag{3.3}$$

$$\mathcal{E}^{i}\epsilon^I - 2i\epsilon^{IJ}\langle\, \not{\mathcal{E}} \mid \mathcal{U}^{*i}\,\rangle\epsilon_J = 0\,, \tag{3.4}$$

where

$$\mathcal{E}^a \equiv \begin{pmatrix} \mathcal{B}^{\Lambda\, a} \\ \mathcal{E}_{\Lambda}{}^{a} \end{pmatrix}. \tag{3.5}$$

We have checked through explicit computation that these relations remain valid in the non-Abelian gauged case at hand.

Taking products of these expressions with Killing spinors and gamma matrices, one can derive KSIs involving the bosonic equations of motion and tensors constructed as bilinears of the commuting Killing spinors.[3] In the case in which the bilinear $V^{\mu} \equiv i\bar{\epsilon}^I\gamma^{\mu}\epsilon_I$ is a timelike vector (referred to as the *timelike case*), one obtains [31] the following identities (w.r.t. an orthonormal frame with $e_0{}^{\mu} \equiv V^{\mu}/|V|$)

$$\mathcal{E}^{ab} = \eta^a{}_0\eta^b{}_0\mathcal{E}^{00}\,, \tag{3.6}$$

$$\langle\, \mathcal{V}/X \mid \mathcal{E}^a\,\rangle = \frac{1}{4}|X|^{-1}\mathcal{E}^{00}\delta^a{}_0\,, \tag{3.7}$$

$$\langle\, \mathcal{U}^*_{i^*} \mid \mathcal{E}^a\,\rangle = \frac{1}{2}e^{-i\alpha}\mathcal{E}_{i^*}\delta^a{}_0\,, \tag{3.8}$$

where $X \equiv \frac{1}{2}\varepsilon_{IJ}\bar{\epsilon}^I\epsilon^J$ and is non-zero in the timelike case, as is paramount from the Fierz identity $V^2 = 4|X|^2$ [9, 12].

As discussed in ref. [31], these identities contain a great deal of physical information. In this paper we shall exploit only one fact, namely the fact that if the Maxwell equation and the Bianchi identity are satisfied for a supersymmetric configuration, then so are the rest of the equations of motion. The strategy to be followed is, therefore, to first identify the supersymmetric configurations and to impose the Maxwell equations and the Bianchi identities. This will lead to some differential equations that need be solved in order to construct a supersymmetric solution.

[3] See the appendix in ref. [12] for the definitions and properties of these bilinears.

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In the case in which V^μ is a null vector (the *null case*), renaming it as l^μ for reasons of clarity, one gets

$$\begin{aligned} \left(\mathcal{E}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \mathcal{E}^\rho{}_\rho\right) l^\nu = \left(\mathcal{E}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \mathcal{E}^\rho{}_\rho\right) m^\nu &= 0\,, && (3.9)\\ \mathcal{E}_{\mu\nu} l^\nu = \mathcal{E}_{\mu\nu} m^\nu &= 0\,, && (3.10)\\ \langle \mathcal{V} \mid \mathcal{E}^\mu \rangle &= 0\,, && (3.11)\\ \langle \mathcal{U}^*_{i^*} \mid \mathcal{E}^\mu \rangle\, l_\mu = \langle \mathcal{U}^*_{i^*} \mid \mathcal{E}^\mu \rangle\, m^*_\mu &= 0\,, && (3.12)\\ \mathcal{E}^i &= 0\,, && (3.13) \end{aligned}$$

where l, n, m, m^* is a null tetrad constructed with the Killing spinor ϵ^I and an auxiliary spinor η as explained in ref. [9].

These identities imply that in the null case the only independent equations of motion that one has to check on supersymmetric configurations are $\mathcal{E}_{\mu\nu} n^\mu n^\nu$ and $\langle \mathcal{U}^*_{i^*} \mid \mathcal{E}_\mu \rangle\, n^\mu$. As before, these are the equations that need to be imposed in order for a supersymmetric configuration to be a supersymmetric solution.

3.2 Killing equations for the bilinears

In order to find the most general background admitting a solution to the KSEs, eqs. (3.1) and (3.2), we shall assume that the background admits one Killing spinor. Using this assumption we will derive consistency conditions that the background must satisfy, after which we will prove that these necessary conditions are also sufficient.

It is convenient to work with spinor bilinears, and consequently we start by deriving equations for these bilinears by contracting the KSEs with gamma matrices and Killing spinors.

From the gravitino supersymmetry transformation rule eq. (2.12) we get the independent equations

$$\begin{aligned} \mathfrak{D}_\mu X &= -i T^+{}_{\mu\nu} V^\nu\,, && (3.14)\\ \mathfrak{D}_\mu V^I{}_{J\,\nu} &= i\delta^I{}_J [X T^{*-}{}_{\mu\nu} - X^* T^+{}_{\mu\nu}] && (3.15)\\ &\quad -i[\epsilon^{IK} T^{*-}{}_{\mu\rho} \Phi_{KJ}{}^\rho{}_\nu - \epsilon_{JK} T^+{}_{\mu\rho} \Phi^{KI\rho}{}_\nu]\,, && (3.16) \end{aligned}$$

which have the same functional form as their equivalents in the ungauged case. Hence, as in the ungauged case, V^μ is a Killing vector and the 1-form $\hat{V} \equiv V_\mu dx^\mu$ satisfies the equation

$$d\hat{V} = 4i[X T^{*-} - X^* T^+]\,. \tag{3.17}$$

The remaining 3 independent 1-forms $\hat{V}^x \equiv \frac{1}{\sqrt{2}} V^I{}_{J\,\mu} \sigma^{x\,J}{}_I dx^\mu$ ($x = 1, 2, 3$ and the σ^x are the Pauli matrices) are exact, i.e.

$$d\hat{V}^x = 0\,. \tag{3.18}$$

From the gauginos' supersymmetry transformation rules, eqs. (2.13), we obtain

$$\begin{aligned} V^I{}_K{}^\mu \mathfrak{D}_\mu Z^i + \epsilon^{IJ} \Phi_{KJ}{}^{\mu\nu} G^{i\,+}{}_{\mu\nu} + W^i \epsilon^{IJ} M_{KJ} &= 0\,, && (3.19)\\ i M^{KI} \mathfrak{D}_\mu Z^i + i \Phi^{KI}{}_\mu{}^\nu \mathfrak{D}_\nu Z^i - 4i\epsilon^{IJ} V^K{}_J{}^\nu G^{i\,+}{}_{\mu\nu} - i W^i \epsilon^{IJ} V^K{}_{J\,\mu} &= 0\,. && (3.20) \end{aligned}$$

The trace of the first equation gives

$$V^{\mu}\mathfrak{D}_{\mu}Z^{i} + 2XW^{i} = 0\,, \tag{3.21}$$

while the antisymmetric part of the second equation gives

$$2X^{*}\mathfrak{D}_{\mu}Z^{i} + 4G^{i\,+}{}_{\mu\nu}V^{\nu} + W^{i}V_{\mu} = 0\,. \tag{3.22}$$

The well-known special geometry completeness relation implies that

$$F^{\Lambda\,+} = i\mathcal{L}^{*\,\Lambda}T^{+} + 2f^{\Lambda}{}_{i}G^{i\,+}\,, \tag{3.23}$$

which allows us to combine eqs. (3.14) and (3.22), as to obtain

$$\begin{aligned} V^{\nu}F^{\Lambda\,+}{}_{\nu\mu} &= i\mathcal{L}^{*\,\Lambda}V^{\nu}T^{+}{}_{\nu\mu} + 2f^{\Lambda}{}_{i}V^{\nu}G^{i\,+}{}_{\nu\mu} \\ &= \mathcal{L}^{*\,\Lambda}\mathfrak{D}_{\mu}X + X^{*}\mathfrak{D}_{\mu}\mathcal{L}^{\Lambda} + \frac{1}{2}W^{i}V_{\mu}\,. \end{aligned} \tag{3.24}$$

Multiplying this equation by V^{μ} and using eq. (3.21), we find

$$V^{\mu}\mathfrak{D}_{\mu}X = 0\,. \tag{3.25}$$

At this point in the investigation, it is convenient to take into account the norm of the Killing vector V^{μ}: we shall investigate the timelike case in section 4 and the null case in section 6.

4. The timelike case

4.1 The vector field strengths

As is well-known, the contraction of the (anti-) self-dual part of a 2-form with a non-null vector, such as V^{μ} in the current timelike case, completely determines the 2-form, i.e.

$$C^{\Lambda\,+}{}_{\mu} \equiv V^{\nu}F^{\Lambda\,+}{}_{\nu\mu} \;\;\Rightarrow\;\; F^{\Lambda\,+} = V^{-2}[\hat{V}\wedge\hat{C}^{\Lambda\,+} + i\,\star(\hat{V}\wedge\hat{C}^{\Lambda\,+})]\,. \tag{4.1}$$

As $C^{\Lambda\,+}{}_{\mu}$ is given by eq. (3.24), the vector field strengths are written in terms of the scalars Z^{i}, X and the vector V. Observe that the component of $C^{\Lambda\,+}{}_{\mu}$ proportional to V^{μ} is projected out in this formula: this implies once again that the field strengths have the same functional form as in the ungauged case. The covariant derivatives that appear in the r.h.s., however, contain explicitly the vector potentials.

The next item on the list is the determination of the spacetime metric:

4.2 The metric

As in the ungauged case we define a time coordinate t by

$$V^{\mu}\partial_{\mu} \;\equiv\; \sqrt{2}\partial_{t} \;\longrightarrow\; \hat{V} \;=\; 2\sqrt{2}\;|X|^{2}\;\left(dt+\hat{\omega}\right)\,, \tag{4.2}$$

where the expression for $\hat{V}$ follows from the Fierz identity $V^{2} = 4|X|^{2}$, which furthermore implies that $\hat{\omega}(\partial_{t}) = 0$.

Unlike the ungauged case, however, the scalars in a supersymmetric configuration need not automatically be time-independent: with respect to the chosen t-coordinate eq. (3.21) takes the form

$$\partial_t Z^i + gA^\Lambda{}_t k_\Lambda{}^i + \sqrt{2}XW^i = \partial_t Z^i + g\Big(A^\Lambda{}_t + \frac{1}{\sqrt{2}}X\mathcal{L}^{*\Lambda}\Big)k_\Lambda{}^i = 0\,. \tag{4.3}$$

It is convenient to choose a G_V gauge in which the complex fields Z^i are time-independent, and one accomplishing just that is

$$A^\Lambda{}_t \;=\; -\sqrt{2}\Re\mathrm{e}\,(X\mathcal{L}^{*\Lambda}) \;=\; -\sqrt{2}|X|^2\Re\mathrm{e}\,(\mathcal{L}^{*\Lambda}/X^*)\,. \tag{4.4}$$

This gauge choice reduces eq. (4.3) to

$$\partial_t Z^i - \frac{1}{\sqrt{2}}gX^*\mathcal{L}^\Lambda k_\Lambda{}^i \;=\; \partial_t Z^i \;=\; 0\,, \tag{4.5}$$

on account of eq. (A.46). It should be pointed out that this gauge choice is identical to the expression for A_t obtained in ungauged case in refs. [9, 11]. Further, using the above t-independence and gauge choice in eq. (3.25), we can derive

$$\begin{aligned}
\partial_t X + i\mathcal{Q}_t X + igA^\Lambda{}_t\mathcal{P}_\Lambda &= \partial_t X + \frac{1}{2}(\partial_t Z^i\partial_i\mathcal{K} - \text{c.c})X + igA^\Lambda{}_t\mathcal{P}_\Lambda X \\
&= \partial_t X - \sqrt{2}ig|X|^2\Re\mathrm{e}\,(\mathcal{L}^{*\Lambda}/X^*)\mathcal{P}_\Lambda X \\
&= \partial_t X = 0\,,
\end{aligned} \tag{4.6}$$

where we made use of eq. (A.45) and the reality of $\mathcal{P}_\Lambda$. Thus, with the standard coordinate choice and the gauge choice (4.4) the scalars Z^i and X are time-independent.

As is shown in refs. [4, 9, 12], the vectorial bilinears in the timelike case, obey an orthogonality relation which allows us to express the metric as

$$ds^2 \;=\; (2|X|)^{-2}\;\Big[\hat{V}\otimes\hat{V} \;-\; 2\;\hat{V}^x\otimes\hat{V}^x\Big] \quad (x = 1,2,3)\;. \tag{4.7}$$

Using, then, the exactness of the 1-forms $\hat{V}^x$ to define spacelike coordinates x^x by

$$\hat{V}^x \;\equiv\; dx^x\,, \tag{4.8}$$

and eq. (4.2), the metric takes on the coordinate form

$$ds^2 \;=\; 2|X|^2(dt+\hat{\omega})^2 - \frac{1}{2|X|^2}dx^x dx^x. \tag{4.9}$$

Furthermore, as V is a Killing vector, $\hat{\omega} = \omega_{\underline{x}}dx^x$ must be a time-independent 1-form. The consistency of eqs. (3.17), (4.1) with the coordinate choice (4.2) implies that this 1-form is determined by the following condition

$$d\hat{\omega} = \frac{i}{2\sqrt{2}}\star\left[\hat{V}\wedge\frac{X\mathfrak{D}X^* - X^*\mathfrak{D}X}{|X|^4}\right] \tag{4.10}$$

Observe that this equation has, apart from a different definition of the covariant derivative, the same functional form as in the ungauged case (See e.g. [9, (4.8)]); before we start rewriting the above result in order to get to the desired result, however, we would like to point out that due to the stationary character of the metric, the resulting covariant derivatives on the transverse $\mathbb{R}^3$ contain a piece proportional to $\omega_{\underline{x}}$. The end-effect of this pull-back is that we introduce a new connection on $\mathbb{R}^3$, denoted by $\tilde{\mathfrak{D}}_{\underline{x}}$, which is formally the same as $\mathfrak{D}_{\underline{x}}$ but for a redefinition of the gauge field, i.e.

$$\tilde{A}^{\Lambda}{}_{\underline{x}} \;=\; A^{\Lambda}{}_{\underline{x}} - \omega_{\underline{x}}\, A^{\Lambda}{}_{t}\,. \tag{4.11}$$

In order to compare the results in this article with the ones found in [9], we introduce the real symplectic sections $\mathcal{I}$ and $\mathcal{R}$ defined by

$$\mathcal{R} \equiv \Re\mathrm{e}(\mathcal{V}/X)\,, \qquad\qquad \mathcal{I} \equiv \Im\mathrm{m}(\mathcal{V}/X)\,. \tag{4.12}$$

Here $\mathcal{V}$ is the symplectic section defining special geometry and therefore satisfies

$$\mathcal{V} = \begin{pmatrix} \mathcal{L}^{\Lambda} \\ \mathcal{M}_{\Sigma} \end{pmatrix}\,, \qquad \langle \mathcal{V} \mid \mathcal{V}^{*}\rangle \equiv \mathcal{L}^{*\,\Lambda}\mathcal{M}_{\Lambda} - \mathcal{L}^{\Lambda}\mathcal{M}^{*}_{\Lambda} = -i\,. \tag{4.13}$$

This implies that our gauge choice can be expressed in the form

$$A^{\Lambda}{}_{t} = -\sqrt{2}|X|^{2}\mathcal{R}^{\Lambda}\,, \tag{4.14}$$

and that the metric function $|X|$ can be written as

$$\frac{1}{2|X|^{2}} = \langle\,\mathcal{R} \mid \mathcal{I}\,\rangle\,. \tag{4.15}$$

Similar to the ungauged case, we can then rewrite eq. (4.10) as

$$(d\hat{\omega})_{xy} = 2\epsilon_{xyz}\langle\,\mathcal{I} \mid \tilde{\mathfrak{D}}_{z}\mathcal{I}\,\rangle\,, \tag{4.16}$$

whose integrability condition reads

$$\langle\,\mathcal{I} \mid \tilde{\mathfrak{D}}_{x}\tilde{\mathfrak{D}}_{x}\mathcal{I}\,\rangle = 0\,, \tag{4.17}$$

and we shall see that, apart from possible singularities [32, 31], the integrability condition is identically satisfied for supersymmetric solutions.

4.3 Solving the Killing spinor equations

In the previous sections we have found that timelike supersymmetric configurations have a metric and vector field strengths given by eqs. (4.9), (3.24) and (4.1) in terms of the scalars X, Z^{i}. It is easy to see that all configurations of this form admit spinors ϵ_{I} that satisfy the Killing spinor equations (3.1), (3.2). The Killing spinors have exactly the same form as in the ungauged case [9]

$$\epsilon_{I} = X^{1/2}\epsilon_{I\,0}\,, \qquad \partial_{\mu}\epsilon_{I\,0} = 0\,, \qquad \epsilon_{I\,0} + i\gamma_{0}\epsilon_{IJ}\epsilon^{J}{}_{0} = 0\,. \tag{4.18}$$

We conclude therefore that we have identified all the supersymmetric configurations of the theory, that belong to the timelike case.

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4.4 Equations of motion

The results of section 3.1 imply that in order to have a classical solution, we only need to impose the Maxwell equations and Bianchi identities on the supersymmetric configurations. In this section, then, we will discuss the differential equations arising from applying the Maxwell and Bianchi equations on the supersymmetric configurations obtained thus far.

As we mentioned in section 4.1 the field strengths of supersymmetric configurations take the same form as in the ungauged case [9] with the Kähler-covariant derivatives replaced by Kähler- and G_V-covariant derivatives. Therefore, the symplectic vector of field strengths and dual field strengths takes the form

$$F = \frac{1}{2|X|^2}\left\{\hat{V}\wedge\mathfrak{D}(|X|^2\mathcal{R}) - \star[\hat{V}\wedge\Im\mathrm{m}(\mathcal{V}^*\mathfrak{D}X + X^*\mathfrak{D}\mathcal{V})]\right\} . \tag{4.19}$$

Operating in the first term we can rewrite it in the form

$$F = -\frac{1}{2}\left\{\mathfrak{D}(\mathcal{R}\hat{V}) - 2\sqrt{2}|X|^2\mathcal{R}d\hat{\omega} + \star\left[\hat{V}\wedge\frac{\Im\mathrm{m}(\mathcal{V}^*\mathfrak{D}X + X^*\mathfrak{D}\mathcal{V})}{|X|^2}\right]\right\} , \tag{4.20}$$

and using the equation of 1-form $\hat{\omega}$, eq. (4.10), which is also identical to that of the ungauged case with the same substitution of covariant derivatives, we arrive at

$$F = -\frac{1}{2}\left\{\mathfrak{D}(\mathcal{R}\hat{V}) + \star(\hat{V}\wedge\mathfrak{D}\mathcal{I})\right\} . \tag{4.21}$$

In what follows we shall use the following Vierbein (e^0, e^x) and the corresponding directional derivatives (θ_0, θ_a), normalized as $e^a(\theta_b) = \delta^a{}_b$:

$$\begin{aligned} e^0 &= \sqrt{2}|X|\,(dt\; +\; \omega)\;, & \theta_0 &= \frac{1}{\sqrt{2}}|X|^{-1}\;\partial_t\;, \\ e^x &= \frac{1}{\sqrt{2}}|X|^{-1}\;dx^{\underline{x}} & \theta_x &= \sqrt{2}|X|\left(\partial_{\underline{x}}\;-\;\omega_{\underline{x}}\partial_t\right)\;. \end{aligned} \tag{4.22}$$

With respect to this basis we have

$$V^\mu\partial_\mu \;=\; 2|X|\;\theta_0\,, \qquad \hat{V} \;=\; 2|X|\;e^0\;, \tag{4.23}$$

and the gauge fixing (4.4) and the constraint (4.3) read

$$A^\Lambda{}_0 \;=\; -|X|\;\mathcal{R}^\Lambda\,, \qquad X^*\;\mathfrak{D}_0 Z^i \;=\; -|X|\;W^i\;. \tag{4.24}$$

The equation that the spacelike components of the field strengths $F^\Lambda{}_{\underline{xy}}$ satisfy can be rewritten in the form

$$\tilde{F}^\Lambda{}_{\underline{xy}} \;=\; -\frac{1}{\sqrt{2}}\epsilon_{xyz}\tilde{\mathfrak{D}}_{\underline{z}}\mathcal{I}^\Lambda\;, \tag{4.25}$$

where the tilde indicates that the gauge field that appears in this equation is the combination $\tilde{A}^\Lambda{}_{\underline{x}}$ defined in eq. (4.11).

This equation is easily recognized as the well-known Bogomol'nyi equation [16] for the connection $\tilde{A}^\Lambda{}_{\underline{x}}$ and the real "Higgs" field $\mathcal{I}^\Lambda$ on $\mathbb{R}^3$. Its integrability condition uses the Bianchi identity for the 3-dimensional gauge connection $\tilde{A}^\Lambda{}_{\underline{x}}$ and, as it turns out, is equivalent to the complete Bianchi identity for the 4-dimensional gauge connection $A^{\Lambda}{}_{\mu}$. It takes the form

$$\tilde{\mathfrak{D}}_{\underline{x}}\tilde{\mathfrak{D}}_{\underline{x}}\, \mathcal{I}^{\Lambda} \;=\; 0\,. \tag{4.26}$$

Taking the Maxwell equation in form notation eq. (2.11) and using heavily the formulae in appendix A we find that all the components are satisfied (as implied by the KSIs) except for one which leads to the equation

$$\tilde{\mathfrak{D}}_{\underline{x}}\tilde{\mathfrak{D}}_{\underline{x}}\mathcal{I}_{\Lambda} \;=\; \frac{1}{2}g^{2}\ \left[f_{\Lambda(\Sigma}{}^{\Gamma}f_{\Delta)\Gamma}{}^{\Omega}\ \mathcal{I}^{\Sigma}\mathcal{I}^{\Delta}\right]\ \mathcal{I}_{\Omega}\,. \tag{4.27}$$

Plugging the above equation and the Bianchi identity (4.26) into the integrability condition for ω, eq. (4.17), leads to

$$\langle\, \mathcal{I}\,|\,\tilde{\mathfrak{D}}_{\underline{x}}\tilde{\mathfrak{D}}_{\underline{x}}\mathcal{I}\,\rangle \;=\; -\mathcal{I}^{\Lambda}\tilde{\mathfrak{D}}_{\underline{x}}\tilde{\mathfrak{D}}_{\underline{x}}\mathcal{I}_{\Lambda} \;=\; -\frac{1}{2}g^{2}f_{\Lambda(\Sigma}{}^{\Gamma}f_{\Delta)\Gamma}{}^{\Omega}\ \mathcal{I}^{\Lambda}\mathcal{I}^{\Sigma}\mathcal{I}^{\Delta}\ \mathcal{I}_{\Omega} \;=\; 0\,, \tag{4.28}$$

which is, ignoring possible singularities, therefore identically satisfied.

4.5 Construction of supersymmetric solutions of $N = 2, d = 4$ SEYM

According to the KSIs, the supersymmetric configurations that satisfy the pair of eqs. (4.26) and (4.27), or equivalently the pair of eqs. (4.25) and (4.27), solve all the equations of motion of the theory. This implies that one can give a step-by-step prescription to construct supersymmetric solutions of any $N = 2, d = 4$ SEYM starting from any solution of the YM-Higgs Bogomol'nyi equations on $\mathbb{R}^3$:

1. Take a solution $\tilde{A}^{\Lambda}{}_{\underline{x}}$, $\mathcal{I}^{\Lambda}$ to the equations

   $$\tilde{F}^{\Lambda}{}_{\underline{xy}} \;=\; -\frac{1}{\sqrt{2}}\epsilon_{xyz}\tilde{\mathfrak{D}}_{\underline{z}}\mathcal{I}^{\Lambda}\,.$$

   As we have stressed repeatedly, these equations are nothing but the YM-Higgs Bogomol'nyi equations on $\mathbb{R}^3$ and there are plenty of solutions available in the literature. However, since in most cases the authors' goal is to obtain regular monopole solutions on $\mathbb{R}^3$, there are many solutions to the same equations that have been discarded because they present singularities. We know, however, that in the Abelian case, the singularities might be hidden by an event horizon.[4] Therefore, we will not require the solutions to the Bogomol'nyi equations to be globally regular on $\mathbb{R}^3$.

2. Given the solution $\tilde{A}^{\Lambda}{}_{\underline{x}}$, $\mathcal{I}^{\Lambda}$, eq. (4.27), which we write here again for the sake of clarity (as we will do with other relevant equations):

   $$\tilde{\mathfrak{D}}_{\underline{x}}\tilde{\mathfrak{D}}_{\underline{x}}\, \mathcal{I}_{\Lambda} \;=\; \frac{1}{2}g^{2}\ \left[f_{\Lambda(\Sigma}{}^{\Gamma}f_{\Delta)\Gamma}{}^{\Omega}\ \mathcal{I}^{\Sigma}\mathcal{I}^{\Delta}\right]\ \mathcal{I}_{\Omega}\,.$$

   becomes a linear equation for the $\mathcal{I}_{\Lambda}$s alone which has to be solved. For compact gauge groups a possible solution is

   $$\mathcal{I}_{\Lambda} = \mathcal{J}\mathcal{I}^{\Lambda}\,, \tag{4.29}$$

   for an arbitrary real constant $\mathcal{J}$ (the r.h.s. of eq. (4.27) vanishes for this Ansatz).

[4] More precisely they turn out to be coordinate singularities in the full spacetime and correspond, not to a singular point, but to an event horizon.

3. The first two steps provide $\mathcal{I} = (\mathcal{I}^\Lambda, \mathcal{I}_\Lambda) = \Im\mathrm{m}\,(\mathcal{V}/X)$. The next step, then, is to obtain $\mathcal{R} = (\mathcal{R}^\Lambda, \mathcal{R}_\Lambda) = \Re\mathrm{e}\,(\mathcal{V}/X)$ as functions of $\mathcal{I}$ by solving the model-dependent *stabilization equations*. The stabilization equations depend only on the specific model one is considering and does not depend on whether the model is gauged or not.

4. Given $\mathcal{R}$ and $\mathcal{I}$, one can compute the metric function $|X|$ using eq. (4.15)
$$\frac{1}{2|X|^2} = \langle\, \mathcal{R} \mid \mathcal{I}\,\rangle\,;$$
the n physical complex scalars Z^i by
$$Z^i \;\equiv\; \frac{\mathcal{L}^i}{\mathcal{L}^0} \;=\; \frac{\mathcal{L}^i/X}{\mathcal{L}^0/X} \;=\; \frac{\mathcal{R}^i + i\mathcal{I}^i}{\mathcal{R}^0 + i\mathcal{I}^0}\,, \tag{4.30}$$
and the metric 1-form $\hat{\omega}$ using eq. (4.16)
$$(d\hat{\omega})_{\underline{xy}} = 2\epsilon_{xyz}\langle\, \mathcal{I} \mid \tilde{\mathfrak{D}}_{\underline{z}}\mathcal{I}\,\rangle\,.$$
This last equation can always be solved locally, as according to eq. (4.28) its integrability equation is solved automatically, at least locally: Since the solutions to the covariant Laplace equations are usually local (they generically have singularities), the integrability condition may fail to be satisfied everywhere, as discussed for example in refs. [32, 33, 31], leading to singularities in the metric. The solution eq. (4.29), however, always leads to exactly vanishing $\hat{\omega}$, whence to static solutions.

   $|X|$ and $\hat{\omega}$ completely determine the metric of the supersymmetric solutions, given in eq. (4.9)
$$ds^2 \;=\; 2|X|^2(dt + \hat{\omega})^2 - \frac{1}{2|X|^2}dx^x dx^x \qquad (x, y = 1, 2, 3)\;.$$

5. Once $\mathcal{I}, \mathcal{R}, |X|$ and $\hat{\omega}$ have been determined, the 4-dimensional gauge potential can be found from eq. (4.14)
$$A^\Lambda{}_t = -\sqrt{2}|X|^2\mathcal{R}^\Lambda\,,$$
and from the definition of $\tilde{A}^\Lambda{}_{\underline{x}}$ eq. (4.11)
$$A^\Lambda{}_{\underline{x}} = \tilde{A}^\Lambda{}_{\underline{x}} + \omega_{\underline{x}}\; A^\Lambda{}_t\;.$$
The procedure we have followed ensures that this is the gauge potential whose field strength is given in eq. (4.21).

In the next section we are going to construct, following this procedure, several solutions.

5. Monopoles and hairy black holes

As we have seen, the starting point in the construction of $N = 2, d = 4$ SEYM supersymmetric solutions is the Bogomol'nyi equation on $\mathbb{R}^3$. Of course, the most interesting solutions to the Bogomol'nyi equations are the monopoles that can be characterised by saying that they are finite energy solutions that are everywhere regular. The fact that the gauge fields are regular does, however, not imply that the full supergravity solution is regular. Indeed, the metric and the physical scalar fields are built out of the "Higgs field", i.e. $\mathcal{I}$, and the precise relations are model dependent and requires knowing the solutions to the stabilization equation.

As the Higgs field in a monopole asymptotes to a non-trivial constant configuration, it asymptotically breaks the gauge group through the Higgs effect. In fact, as we are dealing with supergravity and supersymmetry preserving solutions, monopoles in our setting would have to implement the super-Higgs effect as for example discussed in refs. [34]. If we were to insist on an asymptotic supersymmetric effective action, we would be forced to introduce hypermultiplets in order to fill out massive supermultiplets, but this point will not be pursued in this article.

The Bogomol'nyi equations admit more than just regular solutions, and we shall give families of solutions, labelled by a continuous parameter $s > 0$, having the same asymptotic behaviour as the monopole solutions. As they are singular on $\mathbb{R}^3$, however, we will use them to construct metrics describing the regions outside regular black holes: as will be shown, the members of a given family lead to black holes that are not distinguished by their asymptotic data, such as the moduli or the asymptotic mass, nor by their entropy and as such illustrate the non-applicability of the no-hair theorem to supersymmetric EYM theories. Furthermore, in all examples considered, the attractor mechanisms is at work, meaning that the physical scalars at the horizon and the entropy depend only on the asymptotic charges and not on the moduli nor on the parameter s.

The plan of this section is as follows: in section (5.1) we shall repeat briefly the embedding of the spherically symmetric solutions to the $SO(3)$ Bogomol'nyi equations in the $\overline{\mathbb{CP}}^3$ models. In all but one of these solutions, the asymptotic gauge symmetry breaking is maximal, i.e. the $SO(3)$ gauge symmetry is broken down to $U(1)$. In section (5.2), we will investigate the embedding of solutions that manifest a non-maximal asymptotic symmetry breaking: for this we take E. Weinberg's spherically symmetric $SO(5)$-monopole [35] embedded into $\overline{\mathbb{CP}}^{10}$. This monopole breaks the $SO(5)$ down to $U(2)$ and has the added characteristic that, unlike the 't Hooft-Polyakov monopole, the Higgs field does not vanish at the origin.

An interesting question is whether one can embed monopoles also into more complicated models. This question will be investigated in section 5.3, where we consider gauged "Magic" supergravities.

5.1 Spherically symmetric solutions in $SO(3)$ gauged $\overline{\mathbb{CP}}^3$

Before discussing the solutions we need to make some comments on the model: the model we shall consider in this and the next section is the so-called $\overline{\mathbb{CP}}^n$ model.[5] In this model the metric on the scalar manifold is that of the symmetric space $SU(1,n)/U(n)$ and the prepotential is given by

$$\mathcal{F} \;=\; \frac{1}{4i}\,\eta_{\Lambda\Sigma}\;\mathcal{X}^{\Lambda}\;\mathcal{X}^{\Sigma}\,,\qquad \eta\;=\;\mathrm{diag}\,(\,+\,,\;[-]^n\;)\;, \tag{5.1}$$

which is manifestly $SO(1,n)$ invariant.

The Kähler potential is straightforwardly derived by fixing $\mathcal{X}^0 = 1$ and introducing the notation $\mathcal{X}^i \;=\; Z^i$; this results in

$$e^{-\mathcal{K}} \;=\; |\mathcal{X}^0|^2 \;-\; \sum_{i=1}^{n}\;|\mathcal{X}^i|^2 \;=\; 1 \;-\; \sum_{i=1}^{n}\;|Z^i|^2 \equiv 1 \;-\; |Z|^2\,. \tag{5.2}$$

Observe that this expression for the Kähler potential implies that the Z's are constrained by $0 \leq |Z|^2 < 1$.

As the model is quadratic, the stabilization equations are easily solved and leads to

$$\mathcal{R}_{\Lambda} \;=\; \frac{1}{2}\eta_{\Lambda\Sigma}\,\mathcal{I}^{\Sigma}\quad,\quad \mathcal{R}^{\Lambda} \;=\; -2\eta^{\Lambda\Sigma}\,\mathcal{I}_{\Sigma}\,. \tag{5.3}$$

With this solution to the stabilization equation, we can express the metrical factor, eq. (4.15), in terms of the $\mathcal{I}$ as

$$\frac{1}{2|X|^2} \;=\; \frac{1}{2}\,\eta_{\Lambda\Sigma}\,\mathcal{I}^{\Lambda}\mathcal{I}^{\Sigma} + 2\eta^{\Lambda\Sigma}\,\mathcal{I}_{\Lambda}\mathcal{I}_{\Sigma} \;=\; \frac{1}{2}\,\eta_{\Lambda\Sigma}\,\mathcal{I}^{\Lambda}\mathcal{I}^{\Sigma}\,, \tag{5.4}$$

where in the last step we used the fact that in this article we shall consider only purely magnetic solutions, so that $\mathcal{I}_{\Lambda} = 0$. The fact that we choose to consider magnetic embeddings only, implies by means of eq. (4.16) that we will be dealing with static solutions.

In order to finish the discussion of the model, we must discuss the possible gauge groups that can occur in the $\overline{\mathbb{CP}}^n$-models: as we saw at the beginning of this section, these models have a manifest $SO(1,n)$ symmetry, under which the $\mathcal{X}$'s transform as a vector. Furthermore, as we are mostly interested in monopole-like solutions, we shall restrict our attention to compact simple groups, which, as implied by eq. (A.51), must be subgroups of $SO(n)$. In fact, eq. (A.51) and eq. (A.42) make the stronger statement that given a gauge algebra $\mathfrak{g}$, the action of $\mathfrak{g}$ on the $\mathcal{X}$'s must be such that only singlets and the adjoint representation appear. For the $\overline{\mathbb{CP}}^n$-models there is no problem whatsoever as we can choose n to be large enough as to accommodate any Lie algebra. Indeed, as is well-known any compact simple Lie algebra $\mathfrak{g}$ is a subalgebra of $\mathfrak{so}(\dim(\mathfrak{g}))$ and the branching of the latter's vector representation is exactly the adjoint representation of $\mathfrak{g}$.

The simplest possibility, namely the $SO(3)$-gauged model on $\overline{\mathbb{CP}}^3$, will be used in the remainder of this section, and the $SO(5)$-gauged $\overline{\mathbb{CP}}^{10}$ model will be used in section (5.2). The $SO(4)$- and the $SU(3)$-gauged models will not be treated, but solutions to these models can be created with great ease using the information in this section and appendix C.

[5]The solutions in this and the next section can also be embedded into the $\mathcal{ST}$-models, with similar conclusions. Contrary to ref. [17], however, we have chosen not to deal with this model explicitly, and refer the reader to appendix B for more details.

As we are restricting ourselves to purely magnetic solutions, which are automatically static, the construction of explicit supergravity solutions goes through the explicit solutions to the $SO(3)$ Bogomol'nyi equation (4.25). Having applications to the attractor mechanism in mind, and being fully aware of the fact that this class consists of only the tip of the iceberg of solutions, we shall restrict ourselves to spherically symmetric solutions to the Bogomol'nyi equations.

Working in gauge theories opens up the possibility of compensating the spacetime rotations with gauge transformations, and in the case of an $SO(3)$ gauge group this means that the gauge connection and the Higgs field, $\mathcal{I}$, after a suitable gauge fixing, takes on the form (See *e.g.* [36])

$$A^{i}{}_{m} \;=\; -\varepsilon_{mn}{}^{i}\, x^{n}\, P(r) \quad , \quad \mathcal{I}^{i} \;=\; -\sqrt{2}\, x^{i}\, H(r)\,. \tag{5.5}$$

Substituting this Ansatz into the Bogomol'nyi equation we find that H and P must satisfy

$$r\partial_r\,(H+P) = gr^2\; P\;\,(H+P)\;, \tag{5.6}$$

$$r\partial_r P \;+\; 2P = H\;\left(1+gr^2P\right)\;. \tag{5.7}$$

All the solutions to the above equations were found in ref. [37] and all but one of them contain singularities. Furthermore, not all of them have the correct asymptotics to lead to asymptotic flat spaces and only part of the ones that do can be used to construct regular supergravity solutions [17, 18]. Here, by a regular supergravity solution we mean that the solution is either free of singularities, which is what is meant by a globally regular solution, or has a singularity but, like the black hole solutions in the Abelian theories, has the interpretation of describing the physics outside the event horizon of a regular black hole. The criterion for this last to occur is that the geometry near the singularity is that of a Robinson-Bertotti/$aDS_2 \times S^2$ spacetime, implying that the black hole has a non-vanishing horizon area, whence also entropy.

The suitable solutions, then, break up into 3 classes:

(I) 't Hooft-Polyakov monopole

This is the most famous solution and reads

$$H \;=\; -\frac{\mu}{gr}\left[\coth(\mu r) - \frac{1}{\mu r}\right] \;\equiv\; -\frac{\mu}{gr}\;\overline{H}(r)\;\;,\;\; P \;=\; -\frac{1}{gr^2}\left[1 \;-\; \mu r\sinh^{-1}(\mu r)\right]\;, \tag{5.8}$$

where μ is a positive constant. The renowned regularity of the 't Hooft-Polyakov monopole opens up the possibility of creating a globally regular solution to the supergravity equations which is in fact trivial to achieve: for the moment we have been ignoring $\mathcal{I}^0$, which, since it is uncharged under the gauge group, is just a real, spherically symmetric harmonic function which we can parametrize as

$$\mathcal{I}^0 = \sqrt{2}(h + p/r)\,. \tag{5.9}$$

It is clear, however, that if we want to avoid singularities, we must take $p = 0$, so that the only free parameter is h.

Let us then discuss the regularity conditions imposed by the metric: as was said before, the solutions are automatically static, so that if singularities in the metric are to appear, they arise from the metrical factor $|X|^2$. Plugging the solution for the Higgs field into the expression (5.4), we find

$$\frac{1}{2|X|^2} \;=\; h^2 \;-\; \frac{\mu^2}{g^2}\,\overline{H}^2(r)\;. \tag{5.10}$$

As one can infer from its definition in eq. (5.8), the function $\overline{H}$ is a monotonic, positive semi-definite function on $\mathbb{R}^+$ and vanishes only at $r = 0$, where it behaves as $\overline{H} \sim \mu r/3 + \mathcal{O}(r^2)$; its behaviour for large r is given by $\overline{H} = 1 - 1/(\mu r)$, which means that we should choose h large enough in order to ensure the positivity of the metrical factor. A convenient choice for h is given by imposing that asymptotically we recover the standard Minkowskian metric in spherical coordinates: this condition gives $h^2 = 1 + \mu^2 g^{-2}$ from which we find the final metrical factor and can then also calculate the asymptotic mass, i.e.

$$\frac{1}{2|X|^2} \;=\; 1 \;+\; \frac{\mu^2}{g^2}\left[1 \;-\; \overline{H}^2\right] \;\;\rightarrow\;\; M \;=\; \frac{\mu}{g^2}\;. \tag{5.11}$$

Written in this form, it is paramount that the metric is globally regular and interpolates between two Minkowksi spaces, one at $r = 0$ and one at $r = \infty$.

In order to show that the solution is a globally regular supergravity solution, we should show that the physical scalars are regular. In the $\overline{\mathbb{CP}}^n$-models the scalars are given by (introducing the outward-pointing unit vector $\vec{n} = \vec{x}/r$)

$$Z^i \;\equiv\; \frac{\mathcal{R}^i + i\mathcal{I}^i}{\mathcal{R}^0 + i\mathcal{I}^0} \;=\; \frac{\mathcal{I}^i}{\mathcal{I}^0} \;=\; \frac{\mu}{gh}\,\overline{H}\,n^i\;, \tag{5.12}$$

so that the regularity is obvious. The scalars also respect the bound $0 \leq |Z|^2 < 1$ as can be seen from the fact that the bound corresponds to the positivity of the metrical factor. This regularity of the scalars and that of the spacetime metric are related [31].

(II) Hairy black holes

A generic class of singular solutions is indexed by a free parameter $s > 0$, called the *Protogenov hair*, and can be seen as a deformation of the 't Hooft-Polyakov monopole, i.e.

$$H \;=\; -\frac{\mu}{gr}\left[\coth(\mu r + s) \;-\; \frac{1}{\mu r}\right] \;\equiv\; -\frac{\mu}{gr}\,\overline{H}_s(r)\;\;,\;\; P \;=\; -\frac{1}{gr^2}\left[1 \;-\; \mu r\sinh^{-1}(\mu r + s)\right]\;. \tag{5.13}$$

The effect of introducing the parameter s is to shift the singularity of the cotangent from $r = 0$ to $\mu r = -s$, i.e. outside the domain of r, but leaving unchanged its asymptotic behaviour.[6] This not only means that the function $\overline{H}_s$ vanishes at some $r_s > 0$, but also that it becomes singular at $r = 0$, so that in order to build a regular solution we must have

[6]One can consider the limiting solution for $s \to \infty$, the result of which was called a black hedgehog in ref. [17]. This solution has, apart from not containing hyperbolic functions, no special properties and will not be considered separately.

$p \neq 0$. Using then the general Ansatz for $\mathcal{I}^0$, eq. (5.9), in order to calculate the metrical factor, we find in stead of eq. (5.10)

$$\frac{1}{2|X|^2} = \left(h + \frac{p}{r}\right)^2 - \frac{\mu^2}{g^2}\,\overline{H}_s^2\,. \tag{5.14}$$

As the asymptotic behaviour of $\overline{H}_s$ is the same as the one for the 't Hooft-Polyakov monopole, the condition imposed by asymptotic flatness still is $h^2 = 1 + \mu^2 g^{-2}$. Given this normalization, the asymptotic mass is

$$M = hp + \frac{\mu}{g^2}\,, \tag{5.15}$$

which should be positive for a physical solution. In this respect, we would like to point out that the product hp should be positive as otherwise the metrical factor would become negative or zero, should it coincide with the zero of $\overline{H}_s$, at a finite distance, ruining our interpretation of the metric as describing the outside of a regular black hole. This then implies that the mass is automatically positive. Finally, let us point out that neither the mass nor the modulus h depend on the Protogenov hair parameter s.

The metrical factor is clearly singular at $r = 0$, but given the interpretation of the metric this is not a problem as long as the geometry near $r = 0$, which corresponds to the near horizon geometry of the black hole, is that of an $aDS_2 \times S^2$ space. This is the case if

$$S_{bh} \equiv \lim_{r\to 0} \frac{r^2}{2|X|^2} = p^2 - \frac{1}{g^2}\,, \tag{5.16}$$

is positive and can thence be identified with the entropy of the black hole.

The scalars for this solution are given by

$$Z^i = \frac{\mu}{g}\,\frac{r\overline{H}_s}{p + hr}\,n^i\,, \tag{5.17}$$

whose asymptotic behaviour is the same as for the 't Hooft-Polyakov monopole. Its behaviour near the horizon, i.e. near $r = 0$, is easily calculated to be

$$\lim_{r\to 0} Z^i = -\frac{1}{gp}\,n^i\,, \tag{5.18}$$

and does not depend on the moduli nor on the Protogenov hair, but only on the asymptotic charges. Observe, however, that since $\overline{H}_s = 0$ at some finite $r_s > 0$, there is a 2-sphere outside the horizon at which the scalars vanish, which is not a singularity for the scalars of this model.

(III) Coloured black holes

There is another particular solution to the $SO(3)$ Bogomol'nyi equation that has all the necessary properties, and this solution is given by

$$H = -P = \frac{1}{gr^2}\left[\frac{1}{1+\lambda^2 r}\right]\,. \tag{5.19}$$

This solution has the same $r \to 0$ behaviour as the hairy solutions, but is such that in the asymptotic regime it has no Higgs v.e.v. nor colour charge. Given the foregoing discussion, it is clear that this solution can be used to build a regular black hole solution, and we can and will be brief.

The regularity of the metric goes once again through the judicious election of h and p: the normalization condition implies that $|h| = 1$ which then also implies that the asymptotic mass of the solution is $M = |p|$. It may seem strange that the YM-configuration does not contribute to the mass, but it does so, at least for a regular black hole solution, in an indirect fashion: the condition for a regular horizon is clearly given by eq. (5.16), which implies that $|p| > 1/g$. With these choices then, the scalars Z are regular for $r > 0$ and at the horizon they behave as in eq. (5.18).

5.2 Non-maximal symmetry breaking in $SO(5)$ gauged $\overline{\mathbb{CP}}^{10}$

In ref. [35], E. Weinberg presented an explicit solution for a spherically symmetric monopole solution that breaks the parent $SO(5)$ gauge group down to $U(2)$; in this section we will discuss the embedding of this solution into supergravity and also generalize it to a family of hairy black holes by introducing Protogenov hair.[7]

The starting point of the derivation of Weinberg's monopole is the explicit embedding of an 't Hooft-Polyakov monopole into an $\mathfrak{so}(3)$ subalgebra of $\mathfrak{so}(5)$. In order to make this embedding paramount we take the generators of $\mathfrak{so}(5)$ to be J_i, $\overline{J}_i$ $(i = 1, 2, 3)$ and P_a $(a = 1, \ldots, 4)$. These generators satisfy the following commutation relations

$$
\begin{aligned}
[J_i, J_j] &= \varepsilon_{ijk}\ J_k\,, & [J_i, P_a] &= P_c\ \Sigma_i{}^c{}_a\,,\\
\left[\bar{J}_i, \bar{J}_j\right] &= \varepsilon_{ijk}\ \bar{J}_k\,, & \left[\bar{J}_i, P_a\right] &= P_c\ \overline{\Sigma}_i{}^c{}_a\,,\\
\left[J_i, \bar{J}_j\right] &= 0\,, & [P_a, P_b] &= -2\ J_i\ \Sigma^i_{ab} - 2\ \bar{J}_i\ \overline{\Sigma}^i_{ab}\,,
\end{aligned}
\tag{5.20}
$$

where we have introduced the 't Hooft symbols Σ_i^{ab} and $\overline{\Sigma}_i^{ab}$. The Σ (*resp.* $\overline{\Sigma}$) are self-dual (*resp.* anti-selfdual) 2-forms on $\mathbb{R}^4$ and satisfy the following matrix relations

$$
\begin{aligned}
[\Sigma_i, \Sigma_j] &= \varepsilon_{ijk}\Sigma_k\,, & \left[\overline{\Sigma}_i, \overline{\Sigma}_j\right] &= \varepsilon_{ijk}\overline{\Sigma}_k\,, & \left[\Sigma_i, \overline{\Sigma}_j\right] &= 0\,,\\
\Sigma_i^2 &= -\frac{1}{4}\ 1_4\,, & \overline{\Sigma}_i^2 &= -\frac{1}{4}\ 1_4\,, & \Sigma_{iab}\overline{\Sigma}_j^{ab} &= 0\,.
\end{aligned}
\tag{5.21}
$$

We would like to stress that $\overline{\Sigma}$ is not the complex nor the Hermitean conjugate of Σ.

Following Weinberg we make the following Ansatz for the $\mathfrak{so}(5)$-valued connection and Higgs field, taking T_A $(A = 1, \ldots, 10)$ to be the generators of $\mathfrak{so}(5)$,

$$
\mathtt{A}_m \equiv A^A{}_m\ T_A \;=\; -\varepsilon_{mj}{}^i n^j \left[rP\ J_i \;+\; rB\ \bar{J}_i\right] \;+\; M_m{}^a\ P_a\,, \tag{5.22}
$$

$$
-\frac{1}{\sqrt{2}}\mathtt{I} \equiv -\frac{1}{\sqrt{2}}\mathcal{I}^A\ T_A \;=\; rH\ n^i J_i \;+\; rK\ n^i \bar{J}_i \;+\; \Omega^a\ P_a\,, \tag{5.23}
$$

[7] In ref. [38] the general equations for a spherically symmetric solution to the $SO(5)$ Bogomol'nyi equations were derived. This opens up the possibility of analysing the system along the lines of ref. [37], but for the moment this has not lead to anything new.

where P, B, H and K are functions of r only. M and Ω are determined by the criterion that we have an 't Hooft-Polyakov monopole in some $\mathfrak{so}(3)$-subalgebra, which we take to be generated by the J_i. One way of satisfying this criterion is by choosing

$$M_m{}^a \;=\; F\;\delta^a_m \quad , \quad \Omega^a \;=\; -F\;\delta^{a0}\;, \tag{5.24}$$

which implies that the Bogomol'nyi equation in the J_i sector reduce to eqs. (5.6) and (5.7).

The analysis of the Bogomol'nyi equations in the remaining sectors impose the constraint that $K = -B$ and the differential equations[8]

$$2g\;F^2 = rK' \;+\; 2K \;+\; K(1 - gr^2K)\;, \tag{5.25}$$

$$F' = \frac{1}{2}gr\;F\,[2P \;+\; H \;+\; K]\;. \tag{5.26}$$

The final ingredient, needed for the calculation of the metrical factor, consists of finding an expression for the $SO(5)$-invariant quantity $\mathcal{I}^A\mathcal{I}^A$: this is

$$\frac{1}{2}\;\mathcal{I}^A\mathcal{I}^A \;=\; r^2H^2 \;+\; r^2K^2 \;+\; 2\;F^2\;. \tag{5.27}$$

In conclusion, given a solution to eqs. (5.6), (5.7), (5.25) and (5.26) we can discuss their embedding into the $SO(5)$-gauged $\overline{\mathbb{CP}}^{10}$-model by means of eq. (5.27).

Weinberg's monopole in supergravity

The explicit form of Weinberg's monopole is given by the solution in eq. (5.8) and

$$K(r) \;=\; -P(r)\;L(r;a) \;\equiv\; \frac{\mu}{gr}\;\overline{K}\;, \tag{5.28}$$

$$F(r) \;=\; \frac{\mu}{2g\;\cosh\left(\mu r/2\right)}\;L^{1/2}(r;a) \;\equiv\; \frac{\mu}{g}\;\overline{F}\;, \tag{5.29}$$

where the profile function L, given by

$$L(r;a) \;=\; \left[1 \;+\; \frac{\mu r}{2a}\coth\left(\mu r/2\right)\right]^{-1}\;, \tag{5.30}$$

depends on a positive parameter a called the *cloud parameter*. The cloud parameter a is a measure for the extension of the region in which the Higgs field in the $\overline{J}_i$- and the P_a-directions are active: in fact when $a = 0$ the profile functions vanishes identically and we are dealing with an embedding of the 't Hooft-Polyakov monopole. The maximal extension is for $a \to \infty$ which then means that $L = 1$.

As one can see from the definitions, K and F are positive semi-definite functions that asymptote exponentially to zero. This not only means that the gauge symmetry is asymptotically broken to $U(2)$, but also that K and F will not contribute to the asymptotic mass, nor to the normalization condition. Unlike the 't Hooft-Polyakov monopole or the

[8] In order to go from Weinberg's notation [35] to ours one needs to change $A \to -rP$, $G \to -rB$, $H \to rH$, $K \to rK$, $e \to -g$ and also $F \to F/\sqrt{2}$.

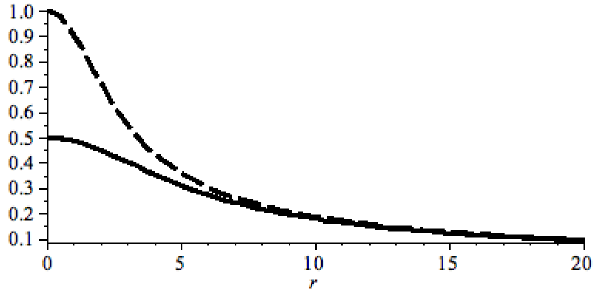

Figure 1: A plot of $1 - \overline{H}^2 - \overline{K}^2 - 2\overline{F}^2$: the dashed line corresponds to $a = 0$ and the solid line corresponds to the maximal cloud extention, i.e. $L = 1$.

degenerate Wilkinson-Bais $SU(3)$-monopole (C.11), however, the regularity of the solution does *not* imply that the Higgs field vanishes at $r = 0$! In fact, near $r = 0$ one finds that

$$\overline{F} \sim \frac{1}{2}\sqrt{\frac{a}{1+a}} + \ldots \quad , \quad \overline{K} \sim \frac{\mu a}{3!(a+1)}\, r + \ldots \, . \tag{5.31}$$

It is this behaviour that may pose a problem for creating a globally regular solution and is the reason for including it in this article.

Using eqs. (5.4) and (5.27) and choosing as in section (5.1) $p = 0$, we can write the metrical factor as

$$\frac{1}{2|X|^2} = 1 + \frac{\mu^2}{g^2}\left[1 - \overline{H}^2 - \overline{K}^2 - 2\overline{F}^2\right] , \tag{5.32}$$

where we already used the normalization condition $h^2 = 1 + \mu^2 g^{-2}$. As mentioned above, $\overline{K}$ and $\overline{F}$ asymptote exponentially to zero and cannot contribute to the mass, which is the one for the 't Hooft-Polyakov monopole, i.e. $M = \mu g^{-2}$.

Let us then investigate the behaviour of (5.32) at $r = 0$: a simple substitution shows that

$$\left.\frac{1}{2|X|^2}\right|_{r=0} = 1 + \frac{\mu^2}{g^2}\,\frac{2a+1}{2(a+1)} , \tag{5.33}$$

which is always positive so that the non-zero value of the Higgs field at the origin is no obstruction to the construction of a globally regular supergravity solution. The remaining question as far as the global regularity of the solution is concerned, is whether there are values of $r > 0$ for which the metrical factor (5.32) becomes negative. This however never happens as one can see from figure 1 which shows a plot of $1 - \overline{H}^2 - \overline{K}^2 - 2\overline{F}^2$ for the values of $a = 0$ and $a = \infty$.

Another hairy black hole

The introduction of Protogenov hair, i.e. a real and positive parameter s, in Weinberg's monopole solution is trivial and leads to the following solution

$$L_s(r;a) = \left[1 \; + \; \frac{\mu r}{2a}\coth\left(\frac{\mu r + s}{2}\right)\right]^{-1} , \tag{5.34}$$

$$F = \frac{\mu}{g}\,\overline{F}_s \; = \; \frac{\mu}{2g\cosh\left(\frac{\mu r+s}{2}\right)}\; L_s^{1/2} , \tag{5.35}$$

$$K = \frac{\mu}{gr}\,\overline{K}_s \; = \; \frac{\mu}{gr}\left[\frac{1}{\mu r} \; - \; \frac{1}{\sinh(\mu r + s)}\right]\; L_s \; . \tag{5.36}$$

supplemented by the expression for H and P given in eq. (5.13). As far as the limiting cases of this family is concerned, it is clear that Weinberg's monopole is obtained in the limit $s \to 0$; in the limit $s \to \infty$ we find that $F \to 0$ and the solution splits up into the direct sum of an $SO(3)$ black hedgehog, i.e. an $s \to \infty$ limit of (5.13), and an $SO(3)$ coloured black hole, eq. (5.19).

As in the case of the hairy $SO(3)$ black holes, the introduction of the hair parameter s preserves the asymptotic behaviour of Weinberg's monopole and the solution is regular for $r > 0$. This immediately implies that the normalization condition for h once again reads $h^2 = 1 + \mu^2 g^{-2}$ and that the asymptotic mass of this solution is given by eq. (5.15), which is positive with the usual proviso that $hp > 0$.

As in the case of the hairy black holes in the $SO(3)$-gauged $\overline{\mathbb{CP}}^3$-models, the regularity of the metric imposes the constraint that the entropy

$$S_{bh} \; = \; p^2 \; - \; \frac{2}{g^2} , \tag{5.37}$$

be positive. This positivity of the entropy also ensures that the physical scalars stay in their domain of definition at $r = 0$. Indeed, the physical scalars can be compactly written as

$$\mathsf{Z} \; = \; Z^A\, T_A \; = \; \frac{\mu}{g}\left[\frac{r\overline{H}_s}{p + hr}\; n^i J_i \; - \; \frac{r\overline{K}_s}{p + hr}\; n^i\overline{J}_i \; + \; \frac{r\overline{F}_s}{p + hr}\; P_0\right] , \tag{5.38}$$

which are therefore regular for $r > 0$. Their value at $r = 0$ is

$$\mathsf{Z}|_{r=0} \; = \; -\frac{1}{gp}\; n^i\left(J_i \; + \; \overline{J}_i\right) , \tag{5.39}$$

which, as in the case of the $SO(3)$ solution, depend only on the asymptotic charges.

5.3 Non-Abelian solutions in Magic models

In this section we would like to discuss the embeddings of monopole solutions into the gauged Magic supergravity theories. We want to show that it is not always possible to construct, given a prepotential for a theory, a globally regular solution based on a given monopole solution. We would like to stress that this holds for a given prepotential, as the choice of symplectic section for a given gauged model is physical due to the breakdown of symplectic invariance.

To start looking for ways to embed monopoles into gauged magic supergravities, we must discuss first the possible gaugings of the magic models, which boils down to a group theory problem whose outcome is given in table 1, which we are going to explain now.

A	G	H	$\mathsf{G}\circ\mathcal{V}$	$\mathsf{H}\circ\mathcal{X}^0$	$\mathsf{H}\circ\mathcal{X}^i$	$\mathsf{I}_3(\mathcal{X}^i)$	$\max(G)$
$\mathbb{R}$	$Sp(3;\mathbb{R})$	$U(3)$	$\mathbf{14}'$	$\mathbf{1}_{-3}$	$\mathbf{6}_{-1}$	$\det(\mathcal{X})$	
$\mathbb{C}$	$SU(3,3)$	$S[U(3)\otimes U(3)]$	$\mathbf{20}$	$(\mathbf{1},\mathbf{1})_{-3}$	$(\mathbf{3},\overline{\mathbf{3}})_{-1}$	$\det(\mathcal{X})$	$SU(3)_{\text{diag}}$
$\mathbb{Q}$	$SO^*(12)$	$U(6)$	$\mathbf{32}'$	$\mathbf{1}_{-3}$	$\mathbf{15}_{-1}$	$\mathrm{Pf}\,(\mathcal{X})$	$SU(4)$
$\mathbb{O}$	$E_{7(-25)}$	$E_6\otimes SO(2)$	$\mathbf{56}$	$\mathbf{1}_3$	$\mathbf{27}_1$	$\mathrm{Tr}\left([\Omega\mathcal{X}]^3\right)/3!$	

Table 1: List of characteristics of Symmetric Special Geometries; all the names of the representations are the ones used by Slansky [39]. The meaning of the different columns is explained in the main text.

The scalar manifolds of the magic models are based on symmetric coset spaces G/H, which are given in the second and the third column in the table. As the isometry-group of the scalar manifold, which for the magic models is isomorphic to G, acts on the symplectic section defining the model (see appendix A), we should specify under what representation of G it transforms; this representation is given in the column denoted as $\mathsf{G}\circ\mathcal{V}$. The following 2 columns determine how the isotropy subgroup H acts on the complex scalars $Z^i=\mathcal{X}^i/\mathcal{X}^0$; the reason why this is important will be discussed presently.

As we are interested in monopoles, we shall restrict ourselves to *compact gauge groups* G, which implies that $G\subseteq\mathsf{H}$. Moreover, as we restricted ourselves to a specific class of gaugings, i.e. gaugings that satisfy eq. (A.42), we should use a prepotential that is G-invariant. Manifestly H-invariant prepotentials for the magic models were given in ref. [40]. These prepotentials are of the STU-type and have the form

$$\mathcal{F}\left(\mathcal{X}\right) \;=\; \frac{\mathsf{I}_3\left(\mathcal{X}^i\right)}{\mathcal{X}^0}\;, \tag{5.40}$$

where I_3 is a cubic H'-invariant,[9] whose value for the specific magic model can be found in the seventh column of table 1.

Another implication of our choice of possible gauge groups is that we can only consider $G\subseteq\mathsf{H}$ for which the branching of the H-representation of the $\mathcal{X}^i$ to G-representations contains only the adjoint representation and singlets. This is a very restrictive property and the maximal possibilities we found are listed in the last column of table 1.

Having discussed the possible models, we must then start discussing the actual embedding of the magnetic monopoles. The first thing is to solve the stabilization equation to find $\mathcal{R}$ in terms of $\mathcal{I}$. This is a complicated question but luckily a general solution exists and was found by Bates and Denef [33]; this solution uses the fact that the generic entropy functions for these models are known. For our purposes, however, the full machinery is not needed. Instead, we shall consider the simpler setting of embedding a purely magnetic monopole in the matter sector and only turn on an electric component for the graviphoton. This means that we should solve the stabilization equations,

$$\begin{aligned} 0 \;&=\Im\mathrm{m}\,\mathcal{L}^0 \quad,\quad \mathcal{I}_0 = -\Im\mathrm{m}\,\left[\mathsf{I}_3(\mathcal{L}^i)/(\mathcal{L}^0)^2\right]\;,\\ \mathcal{I}^i \;&=\Im\mathrm{m}\,\mathcal{L}^i \quad,\quad 0 \;=\Im\mathrm{m}\,\left[\partial_i\mathsf{I}_3(\mathcal{L}^i)\;/\mathcal{L}^0\right]\;, \end{aligned} \tag{5.41}$$

[9] By H' we mean H minus the $U(1)$-factors.

where we absorbed the function X into the $\mathcal{L}$'s. This system admits a solution

$$\mathcal{R}^i \;=\; 0 \;\;,\;\; \mathcal{R}^0 \;=\; -\frac{\sqrt{\mathcal{I}_0\;\mathsf{I}_3(\mathcal{I}^i)}}{\mathcal{I}_0} \;\;\text{provided that}\;\; \mathcal{I}_0\;\mathsf{I}_3(\mathcal{I}^i) > 0\;. \tag{5.42}$$

With this solution to the stabilization equation, it is then straightforward to use eq. (4.15) to determine

$$\frac{1}{2|X|^2} \;=\; 4\sqrt{\;\mathcal{I}_0\;\mathsf{I}_3(\mathcal{I}^i)\;}\;. \tag{5.43}$$

5.3.1 The $\mathbb{C}$-magic model

Let us then consider the $\mathbb{C}$-magic model, which allows an $SU(3)$ gauging. The reason why this is the case is easy to understand: as one can see from table 1 the $\mathcal{L}$'s transform under $SU(3)\otimes SU(3)$ as a $(\mathbf{1},\mathbf{1})\oplus(\mathbf{3},\overline{\mathbf{3}})$ representation. Choosing to gauge the diagonal $SU(3)$ means identifying the left and the right $SU(3)$ actions so that w.r.t. the diagonal action the $\mathcal{L}$'s transform as $\mathbf{1}\oplus\mathbf{3}\otimes\overline{\mathbf{3}}=\mathbf{1}\oplus\mathbf{1}\oplus\mathbf{8}$, which is just what we wanted.

The spherically symmetric monopole solution to the $SU(3)$ Bogomol'nyi equations were found by Wilkinson and Bais in ref. [41], and a discussion of these solutions is given in appendix C. In order to discuss the embedding of the WB-monopole, we gather the components of the symplectic vector $\mathcal{I}$ into a 3×3 matrix, $\mathcal{I}^{\mathbf{1}\oplus\mathbf{8}}$, and as this matrix behaves as the sum of a singlet and the adjoint under the diagonal $SU(3)$, we must take it to be

$$\mathcal{I}^{\mathbf{1}\oplus\mathbf{8}} \;=\; \frac{1}{\sqrt{2}}\left(\lambda\;\mathbf{I}_3\;-\;2\Phi\right)\;, \tag{5.44}$$

where Φ is defined in eq. (C.2) and

$$\lambda = l + L/r\,, \tag{5.45}$$

is a real and spherically symmetric harmonic function. If we then also conveniently redefine $\sqrt{2}\mathcal{I}_0\equiv H$, where

$$H = h + q/r\,, \tag{5.46}$$

is another real harmonic function, we can express eq. (5.43) as

$$\frac{1}{2|X|^2} \;=\; \sqrt{\;H\;(\lambda\;-\;\phi_1)\;(\lambda\;-\;\phi_2\;+\;\phi_1)(\lambda\;+\;\phi_2)\;}\;. \tag{5.47}$$

Given the asymptotic behaviour of the WB solution, let us for clarity discuss the non-degenerate solution whose asymptotic behaviour is given in eq. (C.10), we can normalize the solution to be asymptotically Minkowski by demanding that

$$1 \;=\; h\;\prod_{a=1}^{3}(l+\mu_a)\;. \tag{5.48}$$

Using this normalization, we can then extract the asymptotic mass which turns out to be

$$M \;=\; \frac{1}{4}\left[\frac{q}{h}\;+\;L\;\sum_{i=1}^{3}(l+\mu_i)^{-1}\;+\;2\frac{\mu_3-\mu_1}{(l+\mu_1)(l+\mu_3)}\right]\;, \tag{5.49}$$

and must be ensured to be positive.

Let us then look for a globally regular embedding of the WB-monopole by tuning the free parameters: as before, we shall take $q = L = 0$ in order to avoid the Coulomb singularities in the Abelian field strengths. The first obvious remark is that h is already fixed in terms of l and the μ_a due to eq. (5.48), so that we need to discuss the possible values for l: a first constraint for l comes from the positivity of the mass. Using the facts that $\mu_1 < 0$ and $\mu_3 > 0$, which follow from the constraint and the chosen ordering, in the mass formula (5.49) we see that this implies

$$M \;=\; \frac{\mu_3 \;-\; \mu_1}{2(l+\mu_1)(l+\mu_3)} \;>\; 0 \quad\Longrightarrow\quad l < -\mu_3 \;\text{ or }\; l > -\mu_1\;. \tag{5.50}$$

As we are interested in finding globally regular embeddings, we should discuss the regularity of the metric at $r = 0$: as the ϕ_i's vanish at the origin we see that regularity implies that

$$h\,l^3 \;=\; \prod_a \left(1+\frac{\mu_a}{l}\right)^{-1} \;>\; 0\;. \tag{5.51}$$

It is not hard to see that the above holds for the 2 bounds on l derived in eq. (5.50). At this point then, the real question is whether, given the constraints on h and l derived above, there are values for r other than $r = 0$ or $r = \infty$ for which the metrical factor in eq. (5.47) vanishes; from the monotonicity of ϕ_1 and ϕ_2 it is clear that if this is to happen, then this is because the factor $\lambda-\phi_2+\phi_1$ vanishes. Seeing, then, that the combination $\phi_1-\phi_2$ takes values between $-\mu_3$ and $-\mu_1$, we see that eq. (5.47) never vanishes if

$$\lambda \;>\; \max\left(|\mu_1|,|\mu_3|\right) \quad\text{or}\quad \lambda \;<\; -\max\left(|\mu_1|,|\mu_3|\right)\;. \tag{5.52}$$

In order to finish the discussion of the regularity, we must have a look at the physical scalars: for the above embedding they are schematically given by $Z^{\mathbf{1}\oplus\mathbf{8}} = i\;\mathcal{I}^{\mathbf{1}\oplus\mathbf{8}}/\mathcal{R}^0$, where $\mathcal{R}^0$ is given in eq. (5.42). The regularity then follows straightforwardly from the regularity of monopole solution and the metric.

5.3.2 The $\mathbb{Q}$-magic model

All the embeddings of YM monopoles discussed till now, share a common ingredient, namely the occurrence of additional Abelian fields, whose associated harmonic functions can be used to compensate for the vanishing of the Higgs field at $r = 0$. In the above example, this rôle is played by λ and $\mathcal{I}_0$ and in the $\overline{\mathbb{CP}}^n$ and $\mathcal{ST}[2,n]$-models by the graviphoton's function $\mathcal{I}^0$: a model in which no such compensator exists is the $\mathbb{Q}$-magic model.

As displayed in table 1, the $\mathcal{X}$ in the matter sector lie in the **15** of $SU(6)$, which corresponds to holomorphic 2-forms. As $SU(6)$ admits an $SO(6) \sim SU(4)$ as a singular subgroup for which the relevant branching is $\mathbf{15} \to \mathbf{15}$, we can try to embed an $SU(4)$ WB monopole [41]. This monopole is given, similar to the $SU(3)$ case, by 3 functions ϕ_i ($i = 1,2,3$) and their embedding into the $\mathbb{Q}$-model has $\mathsf{I}_3(\mathcal{I}) = \mathrm{Pf}(\mathcal{X}) = \phi_1\phi_2\phi_3$. The asymptotic behaviour can of course be compensated for by choosing $\mathcal{I}_0$ judiciously, but the real problem lies at $r = 0$. At the origin the ϕ_i vanish as $\phi_1 \sim r^3$, $\phi_2 \sim r^4$ and $\phi_3 \sim r^3$ [41], which means that at the origin we have $\mathsf{I}_3(\mathcal{I}) \sim r^7 + \ldots$ The only freedom we then have is to use the harmonic function $\mathcal{I}_0$, but it is straightforward to see that this is of no use whatsoever, meaning that the resulting spacetime, as well as the physical scalars, are singular at $r = 0$.

Growing hair on the $SU(3)$ WB-monopole

Let us then end this section with a small discussion of the hairy black hole version of the $SU(3)$-monopole. As is discussed in appendix (C.1), singular deformations of the $SU(3)$-monopole can be found with great ease, and is determined by constants β_a ($a = 1, 2, 3$) whose sum is zero. The hard part is to determine the values for the β's for which the metrical factor (5.47) does not vanish for $r > 0$. In fact, lacking general statements about the behaviour of the ϕ's, or the Q's, for general β, we shall restrict ourselves to the minimal choice $\beta_a = s\mu_a$ for $s > 0$. For this choice of β's, seeing as we are only shifting the position of where the Q's vanish from $r = 0$ to $r = -s$, the Q are monotonic, positive definite functions on $\mathbb{R}^+$. If we then rewrite the ϕ's as

$$\phi_i(r) \;=\; -\partial_r \log(Q_i) + \frac{2}{r} \;=\; -\partial_r \log(Q_i) + \frac{2}{r+s} + \frac{2s}{r(s+r)} \;\equiv\; \varphi_i(r;s) + \frac{2s}{r(s+r)} \;, \quad (5.53)$$

where the φ_i are regular and vanish only at $r = -s$; in fact, they correspond to the monopole's Higgs field, and are therefore negative definite on $\mathbb{R}^+$. As pointed out in the appendix, the asymptotic behaviour of the ϕ_i's remain the same as in the monopole case, so that also the normalization condition (5.48) and the asymptotic mass of the object (5.49) remain the same.

The negativity of the φ_i brings us to the next point, namely the absence of zeroes of the metrical factor at non-zero r. This is best illustrated by having a look at the function H in eq. (5.47): it is clear that if H is to have no zeroes for $r > 0$, then h and q must be either both positive or negative, as otherwise $H = 0$ at $|h|r = |q|$. Following this line of reasoning on all the individual building blocks of the metrical factor in eq. (5.47), and choosing for convenience h and q to be positive, shows that we must take

$$\lambda \;>\; \max(|\mu_1|, |\mu_3|) \quad \text{and} \quad L \;>\; 2 \;, \quad (5.54)$$

which automatically implies that the mass, eq. (5.49), is positive.

In order to show that this solution corresponds to the description of a black hole outside its horizon, we must show that the near origin geometry is that of a Robinson-Bertotti/$AdS_2 \times S^2$ spacetime. As the φ_i are regular at $r = 0$, the singularities in the Higgs field come from the $1/r$ terms in eq. (5.53); it is then easy to see that the near-origin geometry is indeed of the required type and that the resulting black hole horizon has entropy

$$S_{bh} \;=\; \sqrt{q\,L\,(L^2 \;-\; 4)}\;. \quad (5.55)$$

Of course, also in this solution the attractor mechanism is at work as one can see by calculating the values of the scalar fields at $r = 0$, i.e.

$$\lim_{r\to 0} Z^{\mathbf{1}\oplus\mathbf{8}} \;=\; \frac{iq}{2S_{bh}}\,\mathrm{diag}\,(\; L-2,\; L,\; L+2\;)\;. \quad (5.56)$$

6. The null case

In the null case the two spinors ϵ_1, ϵ_2 are proportional and, following the same procedure as in refs. [9, 11], we can write[10] $\epsilon_I = \phi_I \epsilon$, where the ϕ_Is are normalized as $\phi_I \phi^I = 1$, and can be understood as a unit vector selecting a particular direction in $SU(2)$ or, equivalently, in S^3. It is useful to project the equations in the $SU(2)$ directions parallel and perpendicular to ϕ_I: for the fermions supersymmetry transformation rules we obtain the following four equations

$$\phi^I \delta_\epsilon \psi_{I\,\mu} = \tilde{\mathfrak{D}}_\mu \epsilon\,, \tag{6.1}$$

$$\phi_I \delta_\epsilon \lambda^{Ii} = i\,\not{\mathfrak{D}} Z^i \epsilon^*\,, \tag{6.2}$$

$$-\epsilon_{IJ}\phi^I \delta_\epsilon \lambda^{Ji} = [\not{G}^{i\,+} + W^i]\epsilon\,, \tag{6.3}$$

$$-\epsilon^{IJ}\phi_I \delta_\epsilon \psi_{J\,\mu} = T^+{}_{\mu\nu}\gamma^\nu \epsilon^* + \epsilon^{IJ}\phi_I \partial_\mu \phi_J \epsilon\,. \tag{6.4}$$

The first three equations are formally identical to the supersymmetry variations of the gravitino, chiralini and gaugini in a gauged $N = 1, d = 4$ supergravity theory with vanishing superpotential that one would get by projecting out the component $N = 2$ gravitini perpendicular to ϕ_I (last equation). This is no coincidence as we could use the Ansatz $\epsilon_I = \phi_I \epsilon$ to perform a truncation of the $N = 2, d = 4$ theory to an $N = 1, d = 4$ theory.[11] Therefore, the $N = 2$ null case reduces to an equivalent $N = 1$ case modulo some details (the presence of the fourth equation and the covariant derivative $\tilde{\mathfrak{D}}$) that will be discussed later. We shall benefit from this fact by using the results of refs. [13, 14] in our analysis. We can also predict the absence of domain-wall solutions in this case, since they only occur in $N = 1, d = 4$ supergravity for non-vanishing superpotential.

Before proceeding, observe that the covariant derivative acting on the supersymmetry parameter ϵ in $\phi^I \delta_\epsilon \psi_{I\,\mu}$ is defined by

$$\tilde{\mathfrak{D}}_\mu \epsilon \equiv \left\{\nabla_\mu + \frac{i}{2}\tilde{\mathcal{Q}}_\mu\right\}\epsilon\,, \qquad \tilde{\mathcal{Q}}_\mu \equiv \hat{\mathcal{Q}}_\mu + \zeta_\mu\,, \tag{6.5}$$

where

$$\zeta_\mu \equiv -2i\phi^I \partial_\mu \phi_I\,, \tag{6.6}$$

is a real $U(1)$ connection associated to the remaining local $U(1)$ freedom left unfixed by our normalization of ϕ_I. It can be shown, by comparing the integrability equations of the above KSEs with the KSIs as in refs. ([5, 9, 11]), that this connection is flat[12] and can be eliminated by choosing the phase of ϵ appropriately. We will assume that this has been done and will ignore it from now on.

[10]The scalars ϕ_I carry a -1 charge and the spinor ϵ a +1 charge, so ϵ_I is neutral. On the other hand, the ϕ_Is have zero Kähler weight and ϵ has Kähler weight 1/2.

[11]The Ansatz of refs. [42, 43] is recovered for the particular choice $\phi_I = \delta_I{}^1$.

[12]This can be understood as follows: except for ζ_μ, all the objects that appear in the KSEs are related to supergravity fields and, when working out the integrability conditions, they end up being related to the different terms of the different equations of motion. The terms derived from ζ_μ (components of its curvature) are unrelated to any fields and one quickly concludes that they must vanish.

The KSEs in the null case are therefore eqs. (6.1)–(6.4) equalled to zero. To analyze them we add to the system an auxiliary spinor η, with the same chirality as ϵ but with opposite $U(1)$ charges and normalized as

$$\bar{\epsilon}\eta = -\bar{\eta}\epsilon = \frac{1}{2}\,. \tag{6.7}$$

This normalization condition will be preserved *iff* η satisfies

$$\mathfrak{D}_\mu \eta + a_\mu \epsilon = 0\,, \tag{6.8}$$

for some a_μ with $U(1)$ charges -2 times those of ϵ, *i.e.*

$$\mathfrak{D}_\mu a_\nu = (\nabla_\mu - i\hat{\mathcal{Q}}_\mu) a_\nu\,, \tag{6.9}$$

to be determined by the requirement that the integrability conditions of this differential equation be compatible with those of the differential equation for ϵ.

The introduction of η allows for the construction of a null tetrad

$$l_\mu = i\sqrt{2}\bar{\epsilon}^*\gamma_\mu\epsilon\,, \quad n_\mu = i\sqrt{2}\bar{\eta}^*\gamma_\mu\eta\,, \quad m_\mu = i\sqrt{2}\bar{\epsilon}^*\gamma_\mu\eta\,, \quad m^*_\mu = i\sqrt{2}\bar{\epsilon}\gamma_\mu\eta^*\,. \tag{6.10}$$

l and n have vanishing $U(1)$ charges but m (m^*) has charge -1 ($+1$), so that the metric constructed using the tetrad

$$ds^2 = 2\hat{l}\otimes\hat{n} - 2\hat{m}\otimes\hat{m}^*\,, \tag{6.11}$$

is invariant.

The orientation of the null tetrad is important: we choose the complex null tetrad $\{e^u, e^v, e^z, e^{z^*}\} = \{\hat{l}, \hat{n}, \hat{m}, \hat{m}^*\}$ such that

$$\epsilon^{uvzz^*} = \epsilon_{uvzz^*} = +i\,, \qquad \gamma_5 \equiv -i\gamma^0\gamma^1\gamma^2\gamma^3 = -\gamma^{uv}\gamma^{zz^*}\,. \tag{6.12}$$

We can also construct three independent selfdual 2-forms:[13]

$$\begin{aligned}
\Phi^{(1)}{}_{\mu\nu} &= \bar{\epsilon}\gamma_{\mu\nu}\epsilon = 2l_{[\mu}m^*_{\nu]}\,, &(6.13)\\
\Phi^{(2)}{}_{\mu\nu} &= \bar{\eta}\gamma_{\mu\nu}\epsilon = [l_{[\mu}n_{\nu]} + m_{[\mu}m^*_{\nu]}]\,, &(6.14)\\
\Phi^{(3)}{}_{\mu\nu} &= \bar{\eta}\gamma_{\mu\nu}\eta = -2n_{[\mu}m_{\nu]}\,, &(6.15)
\end{aligned}$$

or, in form language

$$\hat{\Phi}^{(1)} = \hat{l}\wedge\hat{m}^*\,, \tag{6.16}$$

$$\hat{\Phi}^{(2)} = \frac{1}{2}[\hat{l}\wedge\hat{n} + \hat{m}\wedge\hat{m}^*]\,, \tag{6.17}$$

$$\hat{\Phi}^{(3)} = -\hat{n}\wedge\hat{m}\,. \tag{6.18}$$

[13]The expression of these 2-forms in terms of the vectors are found by studying the contractions between the 2-forms and vectors using the Fierz identities.

6.1 Killing equations for the vector bilinears and first consequences

Let us first consider the algebraic KSEs eqs. (6.2)–(6.4): from them one can immediately obtain

$$
\begin{aligned}
\mathfrak{D}Z^i &= = A^i\hat{l} + B^i\hat{m}\,, && (6.19)\\
T^+ &= \frac{1}{2}\phi\,\hat{\Phi}^{(1)}\,, && (6.20)\\
G^{i\,+} &= \frac{1}{2}\phi^i\,\hat{\Phi}^{(1)} - \frac{1}{2}W^i\hat{\Phi}^{(2)}\,, && (6.21)\\
\epsilon^{IJ}\phi_I d\phi_J &= \frac{i}{\sqrt{2}}\phi\hat{l}\,, && (6.22)
\end{aligned}
$$

where ϕ, ϕ^i, A^i and B^i are complex functions to be determined.

The last equation combined with the vanishing of ζ_μ imply that

$$d\phi_I \sim \hat{l}\,, \qquad d\phi \sim \hat{l}\,. \tag{6.23}$$

The resulting vector field strengths $F^{\Lambda\,+}$ are of the form

$$F^{\Lambda\,+} = \frac{1}{2}\,\phi^\Lambda\hat{\Phi}^{(1)} - \frac{i}{2}\mathcal{D}^\Lambda\,\hat{\Phi}^{(2)}\,, \tag{6.24}$$

where the ϕ^Λ are complex functions related to ϕ and ϕ^i by

$$\phi^\Lambda = i\mathcal{L}^{*\Lambda}\phi + 2f^\Lambda{}_i\phi^i\,, \tag{6.25}$$

and we have defined

$$\mathcal{D}^\Lambda \;\equiv\; -2if^\Lambda{}_iW^i\,. \tag{6.26}$$

Observe that as

$$\mathcal{D}^\Lambda = -igf_{\Sigma\Omega}{}^\Lambda\mathcal{L}^\Omega\mathcal{L}^{*\Sigma} = \frac{1}{2}g\Im\mathrm{m}\mathcal{N}^{-1|\Lambda\Sigma}\mathcal{P}_\Sigma \tag{6.27}$$

is real, we find that the field strengths are given by

$$F^\Lambda = -\frac{1}{2}(\phi^{*\Lambda}\hat{m} + \phi^\Lambda\hat{m}^*)\wedge\hat{l} - \frac{i}{2}\mathcal{D}^\Lambda\,\hat{m}\wedge\hat{m}^*\,. \tag{6.28}$$

Let us consider the differential KSE $\mathfrak{D}_\mu\epsilon = 0$ and the auxiliar KSE eq. (6.8): a straightforward calculation results in

$$
\begin{aligned}
\mathfrak{D}_\mu l_\nu &= \nabla_\mu l_\nu = 0\,, && (6.29)\\
\mathfrak{D}_\mu n_\nu &= \nabla_\mu n_\nu = -a^*_\mu m_\nu - a_\mu m^*_\nu\,, && (6.30)\\
\mathfrak{D}_\mu m_\nu &= (\nabla_\mu - i\hat{\mathcal{Q}}_\mu)m_\nu = -a_\mu l_\nu\,. && (6.31)
\end{aligned}
$$

The first of these equations implies that l^μ is a covariantly constant null Killing vector, eq. (6.29), which tells us that the spacetime is a Brinkmann *pp*-wave [44]. Since l^μ is a Killing vector and $d\hat{l} = 0$ we can introduce the coordinates u and v such that

$$
\begin{aligned}
\hat{l} = l_\mu dx^\mu &\equiv du\,, && (6.32)\\
l^\mu\partial_\mu &\equiv \frac{\partial}{\partial v}\,. && (6.33)
\end{aligned}
$$

We can also define a complex coordinate z by

$$\hat{m} = e^U dz\,, \tag{6.34}$$

where U may depend on z, z^* and u but not on v. Given the chosen coordinates, the most general form of $\hat{n}$ is

$$\hat{n} = dv + Hdu + \hat{\omega}\,, \qquad \hat{\omega} = \omega_{\underline{z}} dz + \omega_{\underline{z}^*} dz^*\,, \tag{6.35}$$

where all the functions in the metric are independent of v. Either H or the 1-form $\hat{\omega}$ could, in principle, be removed by a coordinate transformation, but we have to check that the tetrad integrability equations (6.29)–(6.31) are satisfied by our choices of e^U, H and $\hat{\omega}$.

Given the above choice of coordinates, eq. (6.11) leads to the metric

$$ds^2 = 2du(dv + Hdu + \hat{\omega}) - 2e^{2U} dzdz^*\,. \tag{6.36}$$

Let us then consider the tetrad integrability equations (6.29)–(6.31): the first equation is solved because the metric does not depend on v. The third equation, with the choice (6.34) for the coordinate z, implies

$$\hat{a} = n^\mu[\partial_\mu U - i\hat{\mathcal{Q}}_\mu]\hat{m} + D\hat{l}\,, \tag{6.37}$$

$$0 = m^\mu[\partial_\mu U - i\hat{\mathcal{Q}}_\mu]\,, \tag{6.38}$$

$$0 = l^\mu A^\Lambda{}_\mu \Im\mathrm{m}\,\lambda_\Lambda\,, \tag{6.39}$$

where D is a function to be determined. The last equation can be solved by the gauge choice

$$l^\mu A^\Lambda{}_\mu = 0\,. \tag{6.40}$$

In this gauge the complex scalars Z^i are v-independent. The remaining components of the gauge field $A^\Lambda{}_\mu$ are also v-independent as is indicated by the absence of a $\hat{l}\wedge\hat{n}$, $\hat{m}\wedge\hat{n}$ or a $\hat{m}^*\wedge\hat{n}$ term in the vector field strength. This in its turn, implies the v-independence of all the components of the vector field strengths, of the functions ϕ^i and, finally, of A^i and B^i.

The above condition does not completely fix the gauge freedom of the system, since v-independent gauge transformations preserve it. We can use this residual gauge freedom to remove the $A^\Lambda{}_{\underline{u}}$ component of the gauge potential by means of a v-independent gauge transformation. This leaves us with only one complex independent component $A^\Lambda{}_{\underline{z}}(z, z^*, u) = (A^\Lambda{}_{\underline{z}^*})^*$ and

$$F^\Lambda{}_{\underline{uz}} = \partial_{\underline{u}} A^\Lambda{}_{\underline{z}} = \frac{1}{2} e^U \phi^\Lambda\,, \tag{6.41}$$

$$F^\Lambda{}_{\underline{zz}^*} = \partial_{\underline{z}} A^\Lambda{}_{\underline{z}^*} + \frac{1}{2} g f_{\Sigma\Omega}{}^\Lambda A^\Sigma{}_{\underline{z}} A^\Omega{}_{\underline{z}^*} - \mathrm{c.c.} = -\frac{i}{2} e^{2U}\mathcal{D}^\Lambda\,. \tag{6.42}$$

We can then treat $F^\Lambda{}_{\underline{zz}^*} dz\wedge dz^*$ as a 2-dimensional YM field strength on the 2-dimensional space with Hermitean metric $2e^{2U}dzdz^*$, both of them depending on the parameter u. This implies that we can always write

$$F^\Lambda{}_{\underline{zz}^*} = 2i\partial_{\underline{z}}\partial_{\underline{z}^*} Y^\Lambda\,, \tag{6.43}$$

for some real $Y^\Lambda(z, z^*, u)$. In the Abelian, i.e. ungauged, case

$$A^\Lambda{}_{\underline{z}} = -i\partial_{\underline{z}} Y^\Lambda \,. \tag{6.44}$$

Using eq. (A.15) we can express the second of the tetrad conditions, eq. (6.38), as

$$\partial_{\underline{z}^*}(U + \mathcal{K}/2) = -g A^\Lambda{}_{\underline{z}^*} \lambda_\Lambda \,. \tag{6.45}$$

In the ungauged case this equation (and its complex conjugate) can be immediately integrated to give $U = -\mathcal{K}/2 + h(u)$. The function $h(u)$ can be eliminated by a coordinate redefinition that does not change the form of the Brinkmann metric.

In the Abelian case of the pure $N = 1, d = 4$ theory, it is possible to have constant momentum maps (D-terms), as considered in ref. [45], and $\lambda_\Lambda = -i\mathcal{P}_\Lambda$ and eq. (6.44) would lead to

$$\partial_{\underline{z}^*}(U + \mathcal{K}/2 + g Y^\Lambda \mathcal{P}_\Lambda) = 0 \,, \tag{6.46}$$

which is solved by $U = -\mathcal{K}/2 - g Y^\Lambda \mathcal{P}_\Lambda + h(u)$; $h(u)$ can once again be eliminated by a coordinate transformation. In the $N = 2, d = 4$ theory, however, it is not possible to use constant momentum maps to gauge an Abelian symmetry and the situation is slightly more complicated. The integrability condition of eq. (6.45) and its complex conjugate is solved by

$$A^\Lambda{}_{\underline{z}^*} \lambda_\Lambda = \partial_{\underline{z}^*}[R(z, z^*, u) + S^*(z^*, u)] \,, \tag{6.47}$$

where R is a real function and $S(z, u)$ a holomorphic function of z, which then implies

$$U = -\mathcal{K}/2 - g(R + S + S^*) \,. \tag{6.48}$$

Finally, the second tetrad integrability equation (6.30) implies

$$D = e^{-U}(\partial_{\underline{z}^*} H - \dot{\omega}_{\underline{z}^*}) \,, \tag{6.49}$$

$$(d\omega)_{\underline{z}\underline{z}^*} = 2ie^{2U} n^\mu \hat{\mathcal{Q}}_\mu \,, \tag{6.50}$$

whence $\hat{a}$ is given by

$$\hat{a} = \left[\dot{U} - \frac{1}{2} e^{-2U} (d\omega)_{\underline{z}\underline{z}^*}\right] \hat{m} + e^{-U}(\partial_{\underline{z}^*} H - \dot{\omega}_{\underline{z}^*}) \hat{l} \,. \tag{6.51}$$

6.2 Killing spinor equations

In the previous section we have shown that supersymmetric configurations belonging to the null case must necessarily have a metric of the form eq. (6.36), vector field strengths of the form eq. (6.28), and scalar field strengths of the form eq. (6.19); they must further satisfy eqs. (6.22), (6.38) and (6.50) for some $SU(2)$ vector ϕ_I. We now want to show that these conditions are sufficient for a field configuration $\{g_{\mu\nu}, A^\Lambda, F^\Lambda, \mathfrak{D}Z^i\}$ to be supersymmetric.

It takes little to no time to see that all the components of the KSEs are satisfied for constant Killing spinors (in the chosen gauge, frame, etc.) that obey the condition

$$\gamma^u \epsilon^I = 0 \,. \tag{6.52}$$

This constraint, which is equivalent to $\gamma^z\epsilon^I = 0$, together with chirality, imply that the Killing spinors live in a complex 1-dimensional space, whence we can write $\epsilon^I = \xi^I\epsilon = 0$. Up to normalization, solving the KSEs requires that $\xi^I = \phi^I$, where the functions ϕ^I are given as part of the definition of the supersymmetric field configuration. As a result, the supersymmetric configurations of this theory preserve, generically, 1/2 of the 8 supercharges.

Observe that in order to prove the existence of Killing spinors it has not been necessary to impose the integrability conditions of the field strengths, i.e. the Bianchi identities of the vector field strengths etc., *nor* the integrability constraints of eqs. (6.22), (6.38) and (6.50). We are however forced to do so in order to have well-defined field configurations in terms of the fundamental fields $\{g_{\mu\nu}, A^\Lambda, Z^i\}$. We will deal with these integrability conditions and the equations of motion in the next section.

6.3 Supersymmetric null solutions

Let us start by computing the Bianchi identities and Maxwell equations taking the expression for $F^{\Lambda\,+}$ in (6.24) as our starting point. We find

$$\begin{aligned}\mathfrak{D}F^{\Lambda\,+} &= \left\{\frac{1}{2}m^{*\,\mu}\mathfrak{D}_\mu\phi^\Lambda - \frac{i}{4}n^\mu\mathfrak{D}_\mu\mathcal{D}^\Lambda - \frac{i}{2}\mathcal{D}^\Lambda n^\mu[\partial_\mu U - i\hat{\mathcal{Q}}_\mu]\right\}\hat{l}\wedge\hat{m}\wedge\hat{m}^* \\ &\quad + \frac{i}{4}\left\{m^{*\,\mu}\mathfrak{D}_\mu\mathcal{D}^\Lambda\hat{l}\wedge\hat{n}\wedge\hat{m} + \text{c.c.}\right\}\,. \end{aligned} \tag{6.53}$$

Observe that the terms in the second line are purely imaginary, so that

$$\begin{aligned}\star\mathcal{B}^\Lambda &= -2\Re\mathrm{e}\,\mathfrak{D}F^{\Lambda\,+} \\ &= -i\left\{\Im\mathrm{m}(m^{*\,\mu}\mathfrak{D}_\mu\phi^\Lambda) - \frac{1}{2}n^\mu\mathfrak{D}_\mu\mathcal{D}^\Lambda - \mathcal{D}^\Lambda n^\mu\partial_\mu U\right\}\hat{l}\wedge\hat{m}\wedge\hat{m}^*\,. \end{aligned} \tag{6.54}$$

A similar calculation for F_Λ leads to

$$\begin{aligned}-\mathfrak{D}F_\Lambda &= -2\Re\mathrm{e}\,\mathfrak{D}(\mathcal{N}^*_{\Lambda\Sigma}F^{\Sigma\,+}) \\ &= -i\left\{\Im\mathrm{m}\,(m^{*\,\mu}\mathfrak{D}_\mu\phi_\Lambda) - \frac{1}{2}n^\mu\mathfrak{D}_\mu\Re\mathrm{e}\,\mathcal{D}_\Lambda - \Re\mathrm{e}\,\mathcal{D}_\Lambda n^\mu\partial_\mu U - \Im\mathrm{m}\,\mathcal{D}_\Lambda n^\mu\hat{Q}_\mu\right\}\hat{l}\wedge\hat{m}\wedge\hat{m}^* \\ &\quad + \Re\mathrm{e}\left[m^{*\,\mu}\mathfrak{D}_\mu\Im\mathrm{m}\,\mathcal{D}_\Lambda\hat{l}\wedge\hat{n}\wedge\hat{m}\right]\,, \end{aligned} \tag{6.55}$$

where

$$\phi_\Lambda \equiv \mathcal{N}^*_{\Lambda\Sigma}\phi^\Sigma\,, \qquad \mathcal{D}_\Lambda \equiv \mathcal{N}^*_{\Lambda\Sigma}\mathcal{D}^\Sigma\,,\ \Rightarrow \Im\mathrm{m}\,\mathcal{D}_\Lambda = -\frac{1}{2}g\mathcal{P}_\Lambda\,. \tag{6.56}$$

Of course we can also calculate

$$\frac{1}{2}g\star\Re\mathrm{e}\,(k^*_{\Lambda\,i}\mathfrak{D}Z^i) = \frac{i}{2}g\Im\mathrm{m}\,(n^\mu\mathfrak{D}_\mu Z^i\partial_i\mathcal{P}_\Lambda)\hat{l}\wedge\hat{m}\wedge\hat{m}^* + \frac{1}{2}g\Re\mathrm{e}\,[m^{*\mu}\mathfrak{D}_\mu Z^i\partial_i\mathcal{P}_\Lambda\hat{l}\wedge\hat{n}\wedge\hat{m}]\,, \tag{6.57}$$

which means that the Maxwell equation can be expressed as

$$\begin{aligned}\star\mathcal{E}_\Lambda &= -\mathfrak{D}F_\Lambda + \frac{1}{2}g\star\Re\mathrm{e}\,(k^*_{\Lambda\,i}\mathfrak{D}Z^i) \\ &= -i\Big\{\Im\mathrm{m}(m^{*\,\mu}\mathfrak{D}_\mu\phi_\Lambda) - \frac{1}{2}n^\mu\mathfrak{D}_\mu\Re\mathrm{e}\,\mathcal{D}_\Lambda - \Re\mathrm{e}\,\mathcal{D}_\Lambda n^\mu\partial_\mu U \\ &\qquad - \Im\mathrm{m}\,\mathcal{D}_\Lambda n^\mu\hat{Q}_\mu - \frac{1}{2}g\Im\mathrm{m}\,(n^\mu\mathfrak{D}_\mu Z^i\partial_i\mathcal{P}_\Lambda)\Big\}\hat{l}\wedge\hat{m}\wedge\hat{m}^* \end{aligned} \tag{6.58}$$

In concordance with the KSIs, the Maxwell equations and Bianchi identities have only one non-trivial component, wherefore all the KSIs that involve them are automatically satisfied.

Finally, the only non-automatically satisfied component of the Einstein equations is

$$\mathcal{E}_{\underline{uu}} = R_{\underline{uu}} + 2\mathcal{G}_{ij^*}A^i A^{*\,j^*} - 2\Im\mathrm{m}\mathcal{N}_{\Lambda\Sigma}\phi^\Lambda\phi^{*\Sigma} = 0\,. \tag{6.59}$$

Using our coordinate and gauge choices $l^\mu A^\Lambda{}_\mu = A^\Lambda{}_{\underline{v}} = 0$ and $n^\mu A^\Lambda{}_\mu = A^\Lambda{}_{\underline{u}} = 0$, we can rewrite the above Bianchi identities, Maxwell equations and Einstein equation as

$$\Im\mathrm{m}\,\mathfrak{D}_{\underline{z}}(e^U\phi^\Lambda) = -\frac{1}{2}\partial_{\underline{u}}(e^{2U}\mathcal{D}^\Lambda)\,, \tag{6.60}$$

$$\Im\mathrm{m}\,\mathfrak{D}_{\underline{z}}(e^U\phi_\Lambda) = -\frac{1}{2}\partial_{\underline{u}}(e^{2U}\Re\mathrm{e}\,\mathcal{D}_\Lambda) - \frac{1}{2}g\Im\mathrm{m}\,[\partial_{\underline{u}}Z^i e^{\mathcal{K}}\partial_i(e^{-\mathcal{K}}\mathcal{P}_\Lambda)]\,, \tag{6.61}$$

$$\begin{aligned}\partial_{\underline{z}}\partial_{\underline{z}^*}H &= \partial_{\underline{z}}\dot{\omega}_{\underline{z}^*} + e^{2U}\left\{\partial_{\underline{u}} + \left[\dot{U} - \frac{1}{2}e^{-2U}(d\omega)_{\underline{zz}^*}\right]\right\}\left[\dot{U} - \frac{1}{2}e^{-2U}(d\omega)_{\underline{zz}^*}\right] \\ &\quad + e^{2U}\mathcal{G}_{ij^*}(A^iA^{*\,j^*} + 2\phi^i\phi^{*j^*}) + \frac{1}{2}e^{2U}|\phi|^2\,,\end{aligned} \tag{6.62}$$

where we made use of

$$\mathfrak{D}_{\underline{z}^*}(e^U\phi^\Lambda) \equiv \partial_{\underline{z}^*}(e^U\phi^\Lambda) + gf_{\Sigma\Omega}{}^\Lambda A^\Sigma{}_{\underline{z}^*}e^U\phi^\Omega\,, \tag{6.63}$$

$$\mathfrak{D}_{\underline{z}^*}(e^U\phi_\Lambda) \equiv \partial_{\underline{z}^*}(e^U\phi_\Lambda) + gf_{\Lambda\Sigma}{}^\Omega A^\Sigma{}_{\underline{z}^*}e^U\phi^\Omega\,. \tag{6.64}$$

To summarize our results, supersymmetric configurations have vector and scalar field strengths and metric given by eqs. (6.28), (6.19) and (6.36) and must satisfy the first-order differential eqs. (6.50) and (6.45). We must also find ϕ_I and ϕ such that

$$\epsilon^{IJ}\phi_I\partial_{\underline{u}}\phi_J = \frac{i}{\sqrt{2}}\phi\,. \tag{6.65}$$

If a supersymmetric configuration satisfies the second-order differential eqs. (6.60)–(6.62) then it satisfies all the classical equations of motion and is hence a supersymmetric solution.

6.3.1 u-independent supersymmetric null solutions

In the u-independent case the equations that we have to solve simplify considerably. First of all, since the complex scalars Z^i are u-independent, we have $A^i = 0$ and $(d\omega)_{zz^*} = 0$, whence we can take $\hat{\omega} = 0$. Furthermore, $\phi^\Lambda = 0$ (see eq. (6.41)), which implies $\phi = \phi^i = 0$ (see eq. (6.25)) and the constancy of ϕ_I, which is otherwise arbitrary. We need to solve eq. (6.45), which is only possible if its integrability condition eq. (6.47), which we repeat here for clarity,

$$A^\Lambda{}_{\underline{z}^*}\lambda_\Lambda = \partial_{\underline{z}^*}[R(z,z^*,u) + S^*(z^*,u)]\,, \tag{6.66}$$

is satisfied. Then, the solution is

$$U = -\mathcal{K}/2 - g(R + S + S^*)\,. \tag{6.67}$$

We also need to find covariantly-holomorphic functions $Z^i(z,z^*)$ by solving

$$\partial_{\underline{z}^*}Z^i + gA^\Lambda{}_{\underline{z}^*}k_\Lambda{}^i = 0\,, \tag{6.68}$$

which depends, however, strongly on the model.

Finally, the remaining e.o.m. that needs to be solve is the Einstein equation eq. (6.62): in this case it reduces to the 2-dimensional Laplace equation and is solved by real harmonic functions H on $\mathbb{R}^2$.

In spite of the apparent simplicity of this system, we have not been able to find solutions different from those of the ungauged theory.

7. Conclusions and outlook

In this paper we have analyzed the conditions that fields have to satisfy in hyperless $N = 2, d = 4$ gauged supergravity ($N = 2, d = 4$ super-Einstein-Yang-Mills theory) in order to give rise to a supersymmetric solution.

We have presented and analyzed some spherically-symmetric solutions in the timelike class, which describe monopoles and hairy black holes. As the monopole solutions to the Bogomol'nyi equations are regular on $\mathbb{R}^3$, we investigated the question of whether this regularity can be extended to the full supergravity solution, which we called global regularity. This is a tricky question whose answer, perhaps disappointingly yet predictably, is that it depends on the model. As should be clear from the results of section 5, the biggest obstruction to generating globally-regular supergravity solutions out of spherically-symmetric monopoles is also one of its virtues, namely that at the origin the Higgs field vanishes; as long as the model we are using has extra Abelian fields, this 'problem' can be obviated, but otherwise, such as happens in the $SO^*(12)$ model, it is a real showstopper.

The hairy black holes were generated by the introduction of a parameter $s > 0$ called the *Protogenov hair*. The introduction of this parameter in the solutions is straightforward and basically consists of doing a coordinate shift in the exponential parts of the explicit expressions for the gauge connection and the Higgs field. The effect of this coordinate shift w.r.t. the monopole solution is to leave unchanged the asymptotic behaviour of the solution, but to change the behaviour of the solution at the origin. In fact, due to the positivity of s, the singularity is of Coulomb type and opens up the possibility of creating black holes similar to the ones occurring in Abelian theories. The solutions we studied show that the asymptotic data needed to specify an $N = 2$ $d = 4$ sugra black hole (i.e. the asymptotic mass, the moduli and the asymptotic charges) are independent of the parameter s which is, however, needed in order to specify the black hole fully and demonstrates the failure of the no-hair theorem for gravity coupled to YM fields in an explicit and analytic manner.[14] More surprisingly, the hair parameters don't show up in other relevant quantities such as the entropy of the black hole or the attractor values for the scalars at the horizon: a general understanding of why this happens is lacking but needed.

The attractor mechanism that holds for the scalars of the Abelian black holes still works, but in a generalized way: the Higgs field is not gauge-invariant and one can only expect "attraction" up to gauge transformations. Gauge-invariant combinations of the scalar fields do have fixed points on the horizon

[14] There can of course be more hairy parameters than just the Protogenov hair. In fact, the *cloud* parameter a in eqs. (5.30) and (5.34) should also be considered as hair.

The question about the multi-monopoles and the multi-non-Abelian black holes comes quite naturally, not only as their embedding into sugra could defy Israel's theorem: even though there is a humongous literature on the subject of multi-monopoles, most of the solutions are not known in explicit form. The general 2-monopole solution to the $SO(3)$ Bogomol'nyi equation was, after considerable effort, generated by Panagopoulos [46], who however did not publish the explicit solution. The limiting case of the 2 constituents coinciding corresponds to Ward's axisymmetric 2-monopole solution [47], who gives explicit formulae for the Higgs field on the symmetry axis, taken to coincide with the z-axis, and on the $z = 0$ plane. These expressions satisfy the bounds for the regularity of the embedding, but hardly constitute a definite answer. Work in this direction is in progress.

Recently the magic supergravities were obtained from superstring theory by means of an asymmetric orbifold construction in refs. [48] and [49]. It would be interesting if these constructions could be generalized to the gauged models, which would shed more light on the stringy properties of the hairy black holes.

On the other hand, the gauged $N = 2, d = 4$ supergravities that we have considered here are certainly not the most general ones. One could gauge R-symmetry and the isometries of the hyperscalar manifold, should there be one. The gauging of R-symmetry in absence of hyperscalars has been recently studied in ref. [10] and the timelike case has been completely resolved. The next step would be to include hypermultiplets and the most general gauging of the hyperscalar manifold (which includes, in a certain limit, the gauging of R-symmetry) combined with the gaugings considered in this paper. The null case of the $N = 2, d = 4$ SEYM theories considered in this paper was related to gauged $N = 1, d = 4$ supergravity without a superpotential (but with a kinetic matrix equal to the complex conjugate of the $N = 2$ period matrix). In the null case of the most general theory $N = 2, d = 4$ that we can consider, one should recover gauged $N = 1, d = 4$ supergravity with both non-trivial kinetic matrix and superpotential, opening the possibility of having supersymmetric domain-wall solutions in this sector. We hope to present new results in this direction soon [50].

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A. Gauging holomorphic isometries of special Kähler manifolds

In this appendix we will review some basics of the gauging of holomorphic isometries of the special Kähler manifold in $N = 2, d = 4$ supergravities coupled to vector supermultiplets.

We start by assuming that the Hermitean metric $\mathcal{G}_{ij^*}$ admits a set of Killing vectors[15] $\{K_\Lambda = k_\Lambda{}^i\partial_i + k^*_\Lambda{}^{i^*}\partial_{i^*}\}$ satisfying the Lie algebra

$$[K_\Lambda, K_\Sigma] = -f_{\Lambda\Sigma}{}^\Omega K_\Omega\,, \tag{A.1}$$

of the group G_V that we want to gauge.

Hermiticity and the ij and i^*j^* components of the Killing equation imply that the components $k_\Lambda{}^i$ and $k^*_\Lambda{}^{i^*}$ of the Killing vectors are, respectively, holomorphic and anti-holomorphic and satisfy, separately, the above Lie algebra. Once (anti-) holomorphicity is taken into account, the only non-trivial components of the Killing equation are

$$\frac{1}{2}\pounds_\Lambda \mathcal{G}_{ij^*} = \nabla_{i^*}k^*_{\Lambda\,j} + \nabla_j k_{\Lambda\,i^*} = 0\,, \tag{A.2}$$

where $\pounds_\Lambda$ stands for the Lie derivative w.r.t. K_Λ.

The standard σ-model kinetic term $\mathcal{G}_{ij^*}\partial_\mu Z^i\partial^\mu Z^{*j^*}$ is automatically invariant under infinitesimal reparametrizations of the form

$$\delta_\alpha Z^i = \alpha^\Lambda k_\Lambda{}^i\,, \tag{A.3}$$

if the α^Λs are constants: if they are to be allowed to be arbitrary functions of the spacetime coordinates $\alpha^\Lambda(x)$ we need to introduce a covariant derivative using as connection the vector fields present in the theory. The covariant derivative is

$$\mathfrak{D}_\mu Z^i = \partial_\mu Z^i + gA^\Lambda{}_\mu k_\Lambda{}^i\,, \tag{A.4}$$

and transforms as

$$\delta_\alpha \mathfrak{D}_\mu Z^i = \alpha^\Lambda(x)\partial_j k_\Lambda{}^i \mathfrak{D}_\mu Z^j = -\alpha^\Lambda(x)(\pounds_\Lambda - K_\Lambda)\mathfrak{D}_\mu Z^j\,, \tag{A.5}$$

provided that the gauge potentials transform as

$$\delta_\alpha A^\Lambda{}_\mu = -g^{-1}\mathfrak{D}_\mu\alpha^\Lambda \equiv -g^{-1}(\partial_\mu\alpha^\Lambda + gf_{\Sigma\Omega}{}^\Lambda A^\Sigma{}_\mu\alpha^\Omega)\,. \tag{A.6}$$

For any tensor[16] Φ transforming covariantly under gauge transformations, i.e. tranforming as

$$\delta_\alpha\Phi = -\alpha^\Lambda(x)(\pounds_\Lambda - K_\Lambda)\Phi\,, \tag{A.7}$$

the gauge covariant derivative is given by

$$\mathfrak{D}_\mu\Phi = \{\nabla_\mu + \mathfrak{D}_\mu Z^i\Gamma_i + \mathfrak{D}_\mu Z^{*i^*}\Gamma_{i^*} - gA^\Lambda{}_\mu(\pounds_\Lambda - K_\Lambda)\}\Phi\,. \tag{A.8}$$

[15] The index Λ always takes values from 1 to $\bar{n}$, but some (or all) of the Killing vectors may be zero.

[16] Spacetime and target space tensor indices are not explicitly shown.

In particular, on $\mathfrak{D}_\mu Z^i$

$$\begin{aligned}\mathfrak{D}_\mu\mathfrak{D}_\nu Z^i &= \nabla_\mu\mathfrak{D}_\nu Z^i + \Gamma_{jk}{}^i\mathfrak{D}_\mu Z^j\mathfrak{D}_\nu Z^k + gA^\Lambda{}_\mu\partial_j k_\Lambda{}^i\mathfrak{D}_\nu Z^j\,, \qquad (A.9)\\ [\mathfrak{D}_\mu,\mathfrak{D}_\nu]\,Z^i &= gF^\Lambda{}_{\mu\nu}k_\Lambda{}^i\,, \qquad (A.10)\end{aligned}$$

where

$$F^\Lambda{}_{\mu\nu} = 2\partial_{[\mu}A^\Lambda{}_{\nu]} + gf_{\Sigma\Omega}{}^\Lambda A^\Sigma{}_{[\mu}A^\Omega{}_{\nu]}\,, \qquad (A.11)$$

is the gauge field strength and transforms under gauge transformations as

$$\delta_\alpha F^\Lambda{}_{\mu\nu} = -\alpha^\Sigma(x)f_{\Sigma\Omega}{}^\Lambda F^\Omega{}_{\mu\nu}\,. \qquad (A.12)$$

An important case is that of tensors depending only on the spacetime coordinates through the complex scalars Z^i and their complex conjugates so that $\nabla_\mu\Phi = \partial_\mu\Phi = \partial_\mu Z^i\partial_i\Phi + \partial_\mu Z^{*i^*}\partial_{i^*}\Phi$. This can only be true irrespectively of gauge transformations if the tensor Φ is invariant, that is

$$\pounds_\Lambda\Phi = 0\,. \qquad (A.13)$$

The gauge covariant derivative of invariant tensors is always the covariant pullback of the target covariant derivative:

$$\mathfrak{D}_\mu\Phi = \mathfrak{D}_\mu Z^i\nabla_i\Phi + \mathfrak{D}_\mu Z^{*i^*}\nabla_{i^*}\Phi\,. \qquad (A.14)$$

Now, to make the σ-model kinetic gauge invariant it is enough to replace the partial derivatives by covariant derivatives.

In $N = 2, d = 4$ supergravity, however, the scalar manifold is not just Hermitean, but special Kähler, and simple isometries of the metric are not necessarily symmetries of the theory: they must respect the special Kähler structure. Let us first study how the Kähler structure is preserved.

The transformations generated by the Killing vectors will preserve the Kähler structure if they leave the Kähler potential invariant up to Kähler transformations, i.e. for each Killing vector K_Λ

$$\pounds_\Lambda\mathcal{K} \equiv k_\Lambda{}^i\partial_i\mathcal{K} + k^{*}_\Lambda{}^{i^*}\partial_{i^*}\mathcal{K} = \lambda_\Lambda(Z) + \lambda^*_\Lambda(Z^*)\,. \qquad (A.15)$$

From this condition it follows that

$$\pounds_\Lambda\lambda_\Sigma - \pounds_\Sigma\lambda_\Lambda = -f_{\Lambda\Sigma}{}^\Omega\lambda_\Omega\,. \qquad (A.16)$$

On the other hand, the preservation of the Kähler structure implies the conservation of the Kähler 2-form $\mathcal{J}$

$$\pounds_\Lambda\mathcal{J} = 0\,. \qquad (A.17)$$

The closedness of $\mathcal{J}$ implies that $\pounds_\Lambda\mathcal{J} = d(i_{K_\Lambda}\mathcal{J})$ and therefore the preservation of the Kähler structure implies the existence of a set of real 0-forms $\mathcal{P}_\Lambda$ known as *momentum map* such that

$$i_{K_\Lambda}\mathcal{J} = \mathcal{P}_\Lambda\,. \qquad (A.18)$$

A local solution for this equation is provided by

$$i\mathcal{P}_\Lambda = k_\Lambda{}^i \partial_i \mathcal{K} - \lambda_\Lambda \,, \tag{A.19}$$

which, on account of eq. (A.15), is equivalent to

$$i\mathcal{P}_\Lambda = -(k^*_\Lambda{}^{i^*} \partial_{i^*} \mathcal{K} - \lambda^*_\Lambda) \,, \tag{A.20}$$

or

$$\mathcal{P}_\Lambda = i_{K_\Lambda} \mathcal{Q} - \frac{1}{2i}(\lambda_\Lambda - \lambda^*_\Lambda) \,. \tag{A.21}$$

The momentum map can be used as a prepotential from which the Killing vectors can be derived:

$$k_{\Lambda\, i^*} = i\partial_{i^*} \mathcal{P}_\Lambda \,. \tag{A.22}$$

Using eqs. (A.1), (A.15) and (A.16) one finds

$$\pounds_\Lambda \mathcal{P}_\Sigma = 2ik_{[\Lambda}{}^i k^*_{\Sigma]}{}^{j^*} \mathcal{G}_{ij^*} = -f_{\Lambda\Sigma}{}^\Omega \mathcal{P}_\Omega \,. \tag{A.23}$$

The gauge transformation rule for a symplectic section Φ of Kähler weight (p,q) is[17]

$$\delta_\alpha \Phi = -\alpha^\Lambda(x)(\mathbb{L}_\Lambda - K_\Lambda)\Phi \,, \tag{A.24}$$

where $\mathbb{L}_\Lambda$ stands for the symplectic and Kähler-covariant Lie derivative w.r.t. K_Λ and is given by

$$\mathbb{L}_\Lambda \Phi = \left\{ \pounds_\Lambda - \left[\mathcal{S}_\Lambda - \frac{1}{2}(p\lambda_\Lambda + q\lambda^*_\Lambda) \right] \right\} \Phi \,, \tag{A.25}$$

where the $\mathcal{S}_\Lambda$ are $\mathfrak{sp}(2\bar{n})$ matrices that provide a representation of the Lie algebra of the gauge group G_V:

$$[\mathcal{S}_\Lambda, \mathcal{S}_\Sigma] = +f_{\Lambda\Sigma}{}^\Omega \mathcal{S}_\Omega \,. \tag{A.26}$$

The gauge covariant derivative acting on these sections is given by

$$\begin{aligned} \mathfrak{D}_\mu \Phi = \Big\{ & \nabla_\mu + \mathfrak{D}_\mu Z^i \Gamma_i + \mathfrak{D}_\mu Z^{*i^*} \Gamma_{i^*} + \frac{1}{2}(pk_\Lambda{}^i \partial_i \mathcal{K} + qk^*_\Lambda{}^{i^*} \partial_{i^*} \mathcal{K}) \\ & + gA^\Lambda{}_\mu \left[\mathcal{S}_\Lambda + \frac{i}{2}(p-q)\mathcal{P}_\Lambda - (\pounds_\Lambda - K_\Lambda) \right] \Big\} \Phi \,. \end{aligned} \tag{A.27}$$

Invariant sections are those for which

$$\mathbb{L}_\Lambda \Phi = 0 \,, \quad \Rightarrow \quad \pounds_\Lambda \Phi = \left[\mathcal{S}_\Lambda - \frac{1}{2}(p\lambda_\Lambda + q\lambda^*_\Lambda) \right] \Phi \,, \tag{A.28}$$

and their gauge covariant derivatives are, again, the covariant pullbacks of the Kähler-covariant derivatives:

$$\mathfrak{D}_\mu \Phi = \mathfrak{D}_\mu Z^i\, \mathcal{D}_i \Phi + \mathfrak{D}_\mu Z^{*i^*}\, \mathcal{D}_{i^*} \Phi \,. \tag{A.29}$$

[17]Again, spacetime and target space tensor indices are not explicitly shown. Symplectic indices are not shown, either.

By hypothesis (preservation of the special Kähler structure), the canonical weight $(1,-1)$ section $\mathcal{V}$ is an invariant section

$$K_\Lambda \mathcal{V} = \left[\mathcal{S}_\Lambda - \frac{1}{2}(\lambda_\Lambda - \lambda^*_\Lambda)\right]\mathcal{V}\,, \tag{A.30}$$

and its gauge covariant derivative is given by

$$\mathfrak{D}_\mu \mathcal{V} = \mathfrak{D}_\mu Z^i\, \mathcal{D}_i \mathcal{V} = \mathfrak{D}_\mu Z^i\, \mathcal{U}_i\,. \tag{A.31}$$

Using the covariant holomorphicity of $\mathcal{V}$ one can write

$$K_\Lambda \mathcal{V} = k_\Lambda{}^i \mathcal{U}_i - i\mathcal{P}_\Lambda \mathcal{V} - \frac{1}{2}(\lambda_\Lambda - \lambda^*_\Lambda)\mathcal{V}\,, \tag{A.32}$$

and, comparing with eq. (A.30) and taking the symplectic product with $\mathcal{V}^*$, we find another expression for the momentum map

$$\mathcal{P}_\Lambda = \langle\, \mathcal{V}^* \mid \mathcal{S}_\Lambda \mathcal{V}\,\rangle\,, \tag{A.33}$$

which leads, via eq. (A.22), to another expression for the Killing vectors

$$k_\Lambda{}^i = i\partial^i \mathcal{P}_\Lambda = i\langle\, \mathcal{V} \mid \mathcal{S}_\Lambda \mathcal{U}^{*i}\,\rangle\,. \tag{A.34}$$

If we take the symplectic product with $\mathcal{V}$ instead, we get the following condition

$$\langle\, \mathcal{V} \mid \mathcal{S}_\Lambda \mathcal{V}\,\rangle = 0\,. \tag{A.35}$$

Using the same identity and $\mathcal{G}_{ij^*} = -i\langle \mathcal{U}_i \mid \mathcal{U}^*_{j^*}\,\rangle$ one can also show that

$$k_\Lambda{}^i k^*_\Sigma{}^{j^*} \mathcal{G}_{ij^*} = \mathcal{P}_\Lambda \mathcal{P}_\Sigma - i\langle\, \mathcal{S}_\Lambda \mathcal{V} \mid \mathcal{S}_\Sigma \mathcal{V}^*\,\rangle\,. \tag{A.36}$$

It follows that

$$\langle\, \mathcal{S}_{[\Lambda} \mathcal{V} \mid \mathcal{S}_{\Sigma]} \mathcal{V}^*\,\rangle\; = -\frac{1}{2} f_{\Lambda\Sigma}{}^\Omega \mathcal{P}_\Omega. \tag{A.37}$$

The gauge covariant derivative of $\mathcal{U}_i$ is

$$\mathfrak{D}_\mu \mathcal{U}_i = \mathfrak{D}_\mu Z^j\, \mathcal{D}_j \mathcal{U}_i + \mathfrak{D}_\mu Z^{*j^*}\, \mathcal{D}_{j^*} \mathcal{U}_i = i\mathcal{C}_{ijk} \mathcal{U}^{*j} \mathfrak{D}_\mu Z^k + \mathcal{G}_{ij^*} \mathcal{V} \mathfrak{D}_\mu Z^{*j^*}\,. \tag{A.38}$$

On the supersymmetry parameters ϵ_I, which have $(1/2,-1/2)$ weight, we have

$$\mathfrak{D}_\mu \epsilon_I = \left\{\nabla_\mu + \frac{i}{2}\hat{\mathcal{Q}}_\mu\right\}\epsilon_I\,, \tag{A.39}$$

where we have defined

$$\hat{\mathcal{Q}}_\mu \equiv \mathcal{Q}_\mu + gA^\Lambda{}_\mu \mathcal{P}_\Lambda\,. \tag{A.40}$$

The formalism, so far, applies to any group G_V of isometries. However, we will restrict ourselves to those for which the matrices

$$\mathcal{S}_\Lambda = \begin{pmatrix} a_\Lambda{}^\Omega{}_\Sigma & b_\Lambda{}^{\Omega\Sigma} \\ & \\ c_{\Lambda\Omega\Sigma} & d_{\Lambda\Omega}{}^\Sigma \end{pmatrix}\,, \tag{A.41}$$

have $b = c = 0$. The symplectic transformations with $b \neq 0$ are not symmetries of the action and the gauging of symmetries with $c \neq 0$ leads to the presence of complicated Chern-Simons terms in the action. The matrices a and d are

$$a_\Lambda{}^\Omega{}_\Sigma = f_{\Lambda\Sigma}{}^\Omega \,, \qquad d_{\Lambda\Omega}{}^\Sigma = -f_{\Lambda\Omega}{}^\Sigma \,. \tag{A.42}$$

These restrictions lead to additional identities. First, observe that the condition eq. (A.35) takes the form

$$f_{\Lambda\Sigma}{}^\Omega \mathcal{L}^\Sigma \mathcal{M}_\Omega = 0 \,, \tag{A.43}$$

and the covariant derivative of eq. (A.35), i.e. $\langle \mathcal{V} \mid \mathcal{S}_\Lambda \mathcal{U}_i \rangle = 0$, leads to

$$f_{\Lambda\Sigma}{}^\Omega (f^\Sigma{}_i \mathcal{M}_\Omega + h_{\Omega\, i} \mathcal{L}^\Sigma) = 0 \,. \tag{A.44}$$

Then, using eqs. (A.33), (A.34) and eqs. (A.35), (A.43) and (A.44) we find that

$$\mathcal{L}^\Lambda \mathcal{P}_\Lambda = 0 \,, \tag{A.45}$$

$$\mathcal{L}^\Lambda k_\Lambda{}^i = 0 \,, \tag{A.46}$$

$$\mathcal{L}^{*\Lambda} k_\Lambda{}^i = -i f^{*\Lambda\, i} \mathcal{P}_\Lambda \,. \tag{A.47}$$

From the first two equations it follows that

$$\mathcal{L}^\Lambda \lambda_\Lambda = 0 \,. \tag{A.48}$$

Some further equations that can be derived and are extensively used in the calculations throughout the text are explicit versions of eqs. (A.33) and (A.34), i.e.

$$\mathcal{P}_\Lambda = 2 f_{\Lambda\Sigma}{}^\Gamma \Re\mathrm{e}\left(\mathcal{L}^\Sigma \mathcal{M}^*_\Gamma\right) \,, \qquad k_{\Lambda\, i^*} = i f_{\Lambda\Sigma}{}^\Gamma \left(f^{*\Sigma}_{i^*} M_\Gamma + \mathcal{L}^\Sigma h^*_{\Gamma i^*}\right) \,. \tag{A.49}$$

Finally, notice the identity

$$k_{\Lambda\, i^*} \mathfrak{D} Z^{*i^*} - k^*_{\Lambda i} \mathfrak{D} Z^i = i \mathfrak{D} \mathcal{P}_\Lambda = i (d\mathcal{P}_\Lambda + f_{\Lambda\Sigma}{}^\Omega A^\Sigma \mathcal{P}_\Omega) \,. \tag{A.50}$$

The absolutely last comment in this appendix is the following: if we start from the existence of a prepotential $\mathcal{F}(\mathcal{X})$, then eq. (A.35) implies

$$0 = f_{\Lambda\Sigma}{}^\Gamma\, \mathcal{X}^\Sigma \partial_\Gamma\, \mathcal{F} \,, \tag{A.51}$$

the meaning of which is that one can gauge only the invariances of the prepotential. To put it differently: if you want to construct a model, based on a prepotential, having $\mathfrak{g}$ as the gauge algebra, you need to pick a prepotential that is $\mathfrak{g}$-invariant.

B. The $\mathcal{ST}[2,n]$ models

The $\mathcal{ST}[2,n]$ models have as their Kähler geometry the homogeneous space $\frac{SU(1,1)}{U(1)} \times \frac{SO(2,n)}{SO(2)\otimes SO(n)}$, which is of complex-dimension $n+1$, and must therefore be embedded into $Sp(n+1;\mathbb{R})$. As we are mainly interested in the solution to the stabilization equations, which for these models were solved in ref. [51], and also in the gaugeability of the model, it is convenient to start with the parametrization of the symplectic section for which no prepotential exists. One advantage of this parametrization is that the $SO(2,n)$ symmetry is obvious as one can see from

$$\mathcal{V}^T \;=\; \left(\mathcal{L}^\Lambda\,,\; \eta_{\Lambda\Sigma}\,\mathrm{S}\mathcal{L}^\Sigma\right) \;\text{ where }\; \eta = \mathrm{diag}\left([+]^2,[-]^n\right) \;\text{ and }\; \mathcal{L}^T\eta\mathcal{L} \;=\; 0\,, \tag{B.1}$$

where the constraint is necessary to ensure the correct number of degrees of freedom. Also, and for want of a better place to say so, we take the symplectic indices to run over $\Lambda = (\underline{1}, 0, \ldots, n)$.

In order to declutter the solution to the stabilization equation, i.e. $\mathcal{I} = \Im\mathrm{m}\,(\mathcal{V}/X)$, we absorb the X into the $\mathcal{L}$ and introduce the abbreviations $p^\Lambda = \mathcal{I}^\Lambda$ and $q_\Lambda = \mathcal{I}_\Lambda$. If we then also use η to raise and lower the indices, we can write the stabilization equations as

$$2i\;p^\Lambda \;=\; \mathcal{L}^\Lambda - \mathcal{L}^{*\Lambda}\;\;,\;\; 2i\;q^\Lambda \;=\; \mathrm{S}\;\mathcal{L}^\Lambda - \mathrm{S}^*\;\mathcal{L}^{*\Lambda} \;\;\longrightarrow\;\; \mathcal{L}^\Lambda \;=\; \frac{q^\Lambda \;-\; \mathrm{S}^*\;p^\Lambda}{\Im\mathrm{m}\,\mathrm{S}}\,. \tag{B.2}$$

The function S is then easily found by solving the constraint $\mathcal{L}_\Lambda\mathcal{L}^\Lambda = 0$, and gives

$$\mathrm{S} \;=\; \frac{p\cdot q}{p^2} \;-\; i\;\frac{\sqrt{\,p^2q^2 \;-\; (p\cdot q)^2\,}}{p^2}\,, \tag{B.3}$$

so that we have the constraint $p^2q^2 > (p\cdot q)^2$; the sign of $\Im\mathrm{m}\,\mathrm{S}$ is fixed by the positivity of the metrical function, which with the above sign reads

$$\frac{1}{2|X|^2} \;=\; 2\sqrt{\,p^2q^2 \;-\; (p\cdot q)^2\,}\,. \tag{B.4}$$

We would like to stress that this solution is manifestly $SO(2,n)$ (co/in)variant and automatically solves the constraint $\mathcal{L}^T\eta\mathcal{L} = 0$, without any constraints on p^Λ nor on q_Λ.

For our applications, namely the regularity of the embeddings of monopoles and the attractor mechanism, it is important to know the expression of the moduli in terms of $(n+1)$ unconstrained fields, one of which should be S as it corresponds to the axidilaton. This means that we should have n unconstrained fields Z^a $(a = 0, 1, \ldots, n-1)$ and express them in terms of p's and q's.

One way of doing this is through the introduction of so-called Calabi-Visentini coordinates which means that $(a = 1, \ldots, n)$

$$\mathcal{L}^{\underline{1}} \;=\; \frac{1}{2}\,Y^0\,\left(1+\vec{Z}^2\right)\,,\; \mathcal{L}^0 \;=\; \frac{i}{2}\,Y^0\,\left(\vec{Z}^2 - 1\right)\,,\; \mathcal{L}^a \;=\; Y^0\,Z^a\,, \tag{B.5}$$

which after solving for Y^0 means that the scalar fields are given by

$$Z^a \;=\; \frac{q^a \;-\; \mathrm{S}^*\,p^a}{q^{\underline{1}} + iq^0 \;-\; \mathrm{S}^*\,(p^{\underline{1}} + ip^0)}\,, \tag{B.6}$$

and S is given by expression (B.3). Observe that in this parametrization the $SO(n)$ invariance is paramount.

In order to discuss the possible groups that can be gauged in these models, let us recall that a given compact simple Lie algebra $\mathfrak{g}$ of a group G is a subalgebra of $\mathfrak{so}(\dim(\mathfrak{g}))$ and that furthermore the latter's vector representation branches into $\mathfrak{g}$'s adjoint representation. This then implies that in an $\mathcal{ST}[2,n]$-model one can always gauge a group G as long as $n \geq \dim(\mathfrak{g})$.

In section 5 the explicit details are given for the $\overline{\mathbb{CP}}^n$ models, but at least as far as the embedding of the monopoles are concerned, the embedding into the $\mathcal{ST}$-models is similar. In order to show that this is the case, consider the case of a purely magnetic solution, so that $q^a = 0$, and take furthermore $q_0 = p^{\underline{1}} = 0$ and normalize $q_{\underline{1}} = 1$. Using this Ansatz in eq. (B.4) we obtain

$$\frac{1}{2|X|^2} \;=\; 2\sqrt{\;p^2\;} \;=\; 2\sqrt{\;(p^0)^2 \;-\; (p^a)^2\;}\;, \tag{B.7}$$

which, apart from the $\sqrt{\;}$, is just the same expression as obtained in the $\overline{\mathbb{CP}}^n$-models and leads to the same conditions for the global regularity of the metric. Using the same Ansatz in eq. (B.6) for the scalars, one finds

$$Z^a \;=\; -i\frac{\sqrt{p^2}}{p^2 \;+\; p^0\sqrt{p^2}}\; p^a\;. \tag{B.8}$$

This then means that as long as $p^0 > 0$ and p^2 is regular and positive definite, as is the case for the solutions in section (5), the embeddings of the monopoles is a globally regular supergravity solution.

C. The Wilkinson-Bais monopole in $SU(3)$

In ref. [41], Bais and Wilkinson derived analytical expressions for the general spherically symmetric monopoles to the $SU(N)$ Bogomol'nyi equations. In this case we are going to discuss their monopole for the case of $SU(3)$ as it can be embedded into the $\overline{\mathbb{CP}}^8$, $ST[2,8]$ and the $SU(3,3)/S[U(3)\otimes U(3)]$ model.

The derivation is best done using Hermitean generators, which means that we use the definitions

$$\mathfrak{D}\Phi \;=\; d\Phi - i\,[A,\Phi] \;\;,\;\; F \;=\; dA \;-\; i\;A\wedge A\,, \tag{C.1}$$

where A and Φ are in $\mathfrak{su}(3)$'s fundamental representation; for convenience we have taken $g = 1$.

The maximal form of the fields compatible with spherical symmetry are given by

$$\Phi \;=\; \frac{1}{2}\,\mathrm{diag}\left[\phi_1(r)\;;\;\phi_2(r) - \phi_1(r)\;;\;-\phi_2(r)\right]\;, \tag{C.2}$$

$$A \;=\; J_3\;\cos(\theta)d\varphi \;+\; \frac{i}{2}\left[C - C^\dagger\right]\;d\theta \;+\; \frac{1}{2}\left[C + C^\dagger\right]\;\sin(\theta)d\varphi\;, \tag{C.3}$$

where $J_3 = \mathrm{diag}(1;0;-1)$ and C is the real and upper-triangular matrix

$$C \;=\; \begin{pmatrix} 0 & a_1(r) & 0 \\ 0 & 0 & a_2(r) \\ 0 & 0 & 0 \end{pmatrix}\;. \tag{C.4}$$

Plugging the above Ansätze into the Bogomol'nyi equation $\mathfrak{D}\Phi = \star F$, leads to the following equations $(i = 1, 2)$

$$r^2 \partial_r \phi_i \;=\; a_i^2 - 2 \;\;,\;\; 2\partial_r a_1 \;=\; a_1\,(2\phi_1 - \phi_2) \;\;,\;\; 2\partial_r a_2 \;=\; a_2\,(2\phi_2 - \phi_1) \;. \tag{C.5}$$

Following Wilkinson and Bais [41], we solve the equations for the a_i by defining new functions $Q_i(r)$ through

$$\phi_i \;=\; -\partial_r \log Q_i + \frac{2}{r} \;\;,\;\; a_1 \;\equiv\; \frac{r\sqrt{Q_2}}{Q_1} \;\;,\;\; a_2 \;\equiv\; \frac{r\sqrt{Q_1}}{Q_2} \;, \tag{C.6}$$

after which the remaining equations are

$$Q_2 \;=\; \partial_r Q_1 \partial_r Q_1 \;-\; Q_1 \partial_r^2 Q_1 \;\;,\;\; Q_1 \;=\; \partial_r Q_2 \partial_r Q_2 \;-\; Q_2 \partial_r^2 Q_2 \tag{C.7}$$

The solution found by Wilkinson & Bais for $SU(3)$ then given by

$$\left.\begin{array}{l} Q_1 \;=\; \sum_{a=1}^{3}\; A_a\; e^{\mu_a r} \\[2ex] Q_2 \;=\; \sum_{a=1}^{3}\; A_a\; e^{-\mu_a r} \end{array}\right\} \;\longleftarrow\; \left\{\begin{array}{lcl} 0 &=& \sum_{a=1}^{3}\; \mu_a \\[2ex] A_1 &=& -A_2 A_3\;\;(\mu_2 - \mu_3)^2 \\ A_2 &=& -A_3 A_1\;\;(\mu_3 - \mu_1)^2 \\ A_3 &=& -A_1 A_2\;\;(\mu_1 - \mu_2)^2 \end{array}\right. \;. \tag{C.8}$$

The solution to the above equations is

$$A_a \;=\; \prod_{b \neq a} (\mu_a - \mu_b)^{-1} \;. \tag{C.9}$$

Defining the useful quantity $V_n \equiv \sum_{a=1}^{3}\; A_a \mu_a^n$, we can see by direct inspection that $V_0 = V_1 = V_3 = 0$ and that $V_1 = 1$. Using these quantities one can see that around $r = 0$ we have $Q_i \sim r^2/2 + \mathcal{O}(r^3)$, which means that the $\phi_i \sim -V_4/3!\; r + \mathcal{O}(r^2)$, implying that the solution is completely regular on $\mathbb{R}^3$. Furthermore, one can show that the Q are monotonic, positive semi-definite functions on $\mathbb{R}^+$ that vanish only at $r = 0$, at which point also its derivative vanishes. This furthermore implies that the ϕ_i are negative semi-definite functions on $\mathbb{R}^+$.

The asymptotic behaviour of the Higgs field is easily calculated and, choosing $\mu_1 < \mu_2 < \mu_3$, is readily seen to be

$$\lim_{r\to\infty} \Phi \;=\; -\frac{1}{2}\,\mathrm{diag}\,(\;\mu_3\;;\;\mu_2\;;\;\mu_1\;) \;+\; \frac{1}{r}\; J_3 \;+ \ldots \tag{C.10}$$

from which the breaking of $SU(3) \to U(1)^2$ is paramount.

The above solution does not admit the possibility of having degenerate μ's, but as emphasised by Wilkinson & Bais, such a solution can be obtained as a limiting solution. For this, define $\mu_1 = -2$, $\mu_2 = 1 - \delta$ and $\mu_3 = 1 + \delta$, for $\delta > 0$, and calculate the solution. This solution admits a non-singular $\delta \to 0$ limit, which is

$$Q_1 \;=\; \frac{1}{9}\left[e^{-2r} \;+\; (3r - 1)e^{r}\right] \;\;,\;\; Q_2 \;=\; \frac{1}{9}\left[e^{2r} \;-\; (3r + 1)e^{-r}\right] \;. \tag{C.11}$$

The symmetry breaking pattern in this degenerate case is $SU(3) \to U(2)$ as becomes clear from the asymptotic behaviour of the Higgs field, i.e.

$$\lim_{r\to\infty} \Phi \;=\; -\mathtt{Y} \;+\; \frac{1}{r}\;\mathtt{Y} \;\text{ where }\; \mathtt{Y} \;=\; \frac{1}{2}\,\mathrm{diag}\,(1\,,\;1\,,\;-2) \;. \tag{C.12}$$

C.1 A hairy deformation of the W&B monopole

The foregoing derivation of Wilkinson & Bais's monopole was cooked up to give a regular solution, and we would like to have a hairy version of this monopole. This is easily achieved by applying the Protogenov trick, which calls for adding constants in the exponential parts of the monopole fields; in this case, we simply extend the Ansatz for the Q_i's to

$$Q_1 \;=\; \sum_{a=1}^{3} A_a\; e^{\mu_a r+\beta_a} \;\;,\;\; Q_2 \;=\; \sum_{a=1}^{3} A_a\; e^{-\mu_a r-\beta_a}\;, \tag{C.13}$$

and plug it into eq. (C.7). Obviously this leads to a solution if $\sum\;\mu_a = \sum\beta_a = 0$ and A_a is once again given by eq. (C.9). Furthermore, it is clear that the asymptotic behaviour does not change and it is the one in eq. (C.10); what does change is the behaviour of the solution at $r = 0$, which is singular except when $\beta_a = 0$.

Using the above expression we can also create a hairy version of the degenerate monopole: we have to make the same Ansatz as the one used in the derivation of eq. (C.11), and also define $\beta_2 = s+\delta\gamma/3$, $\beta_3 = s-\delta\gamma/3$ and $\beta_1 = -2s$, which is the maximal possibility compatible with a regular limit. Taking then the limit $\delta \to 0$ we find

$$Q_1 \;=\; \frac{1}{9}\left[e^{-2(r+s)} \;+\; (3r+\gamma-1)e^{r+s}\right] \;\;,\;\; Q_2 \;=\; \frac{1}{9}\left[e^{2(r+s)} \;-\; (3r+\gamma+1)e^{-(r+s)}\right]\;. \tag{C.14}$$

which leads to ϕ_i's that are singular at $r = 0$ but with the asymptotic behaviour displayed in eq. (C.12).

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